

Newton polyhedra of discriminants of projections.

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For a system of polynomial equations, whose coefficients depend on parameters, the Newton polyhedron of its discriminant is computed in terms of the Newton polyhedra of the coefficients. This leads to an explicit formula (involving Euler obstructions of toric varieties) in the unmixed case, suggests certain open questions in general, and generalizes similar known results ([GKZ], [S94], [McD], [G], [EKh]).

Introduction.

Let F_0, \dots, F_l be Laurent polynomials on the complex torus $(\mathbb{C} \setminus 0)^k$, whose coefficients are Laurent polynomials on the parameter space $(\mathbb{C} \setminus 0)^n$. Consider the set $\Sigma \subset (\mathbb{C} \setminus 0)^n$ of all values of the parameter, such that the corresponding system of polynomial equations $F_0 = \dots = F_l = 0$ defines a singular set in $(\mathbb{C} \setminus 0)^k$. In most cases (see below for details), the closure of Σ is a hypersurface, and its defining equation is called the *discriminant* of $F_0 = \dots = F_l = 0$.

In this paper, we compute the Newton polyhedron of the discriminant in terms of Newton polyhedra of the coefficients of the polynomials F_0, \dots, F_l . The answer is known in many special cases, and we give a number of references as examples of various approaches to this problem: the universal special case for $l = 0$ and $l = k$ was studied in [GKZ], [S94] and [DFS] (*universal case* means that $(\mathbb{C} \setminus 0)^n$ parameterizes all collections of polynomials F_0, \dots, F_l , whose monomials are contained in a given finite set of monomials), the general case for $l = k - 1$, for $l = k - 1 = 0$ and for $l = k$ was studied in [McD], [G] and [EKh].

To formulate the answer in general, we need the following notation: we denote the Minkowski sum $\{a + b \mid a \in A, b \in B\}$ of polyhedra A and B by $A + B$, and denote the mixed fiber polyhedron of the polyhedra $\Delta_0, \dots, \Delta_l$ in $\mathbb{R}^n \oplus \mathbb{R}^k$ by the monomial $\Delta_0 \cdot \dots \cdot \Delta_l$ (it is a certain polyhedron in \mathbb{R}^k , see [McM], [EKh], Definition 1.11, or Appendix). To a set $A \subset \mathbb{Z}^k$ and a face B of its convex hull we associate its Euler obstruction $e^{B,A} \in \mathbb{Z}$, whose combinatorial definition is given in Subection 1.5, and whose geometrical meaning is $(-1)^{k-\dim B}$ times the Euler obstruction of the A -toric variety at its orbit, corresponding to B (see also [MT] and a remark at the end of Subsection 4.4).

Considering F_i as a polynomial on $(\mathbb{C} \setminus 0)^k$ with polynomial coefficients, denote the set of its monomials by $A_i \subset \mathbb{R}^k$; considering the same F_i as a polynomial on $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k$ with complex coefficients, denote its Newton polyhedron by $\Delta_i \subset \mathbb{R}^n \oplus \mathbb{R}^k$. Denote the preimage of a set A' under the natural projection $\Delta_i \rightarrow (\text{convex hull of } A_i)$ by $\Delta_i(A')$. For simplicity, we assume here that $A_0 = \dots = A_l = A$, and pairwise differences of its points generate \mathbb{Z}^k .

THEOREM. *If F_0, \dots, F_l are generic polynomials with Newton polyhedra $\Delta_0, \dots, \Delta_l$, then the Newton polyhedron of the discriminant equals*

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$$\mathcal{N} = \sum_{A' \subset A} e^{A', A} \cdot \sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = \dim A' + 1}} \Delta_0(A')^{a_0} \cdot \dots \cdot \Delta_l(A')^{a_l},$$

where A' runs over all faces of dimension l and greater in the convex hull of A .

More precisely, if the polyhedron \mathcal{N} consists of one point, then the discriminant set Σ has no codimension 1 components; otherwise, the closure of Σ is a hypersurface, and the Newton polyhedron of its equation equals \mathcal{N} .

The word “generic” means that the set of collections F_0, \dots, F_l , satisfying the statement, is dense in the space of collections with the given Newton polyhedra $\Delta_0, \dots, \Delta_l$. Note that coefficients $e^{A', A}$ may be negative, and the formula above involves subtraction of polyhedra. The difference of polyhedra P and Q is by definition the solution of the equation $Q + X = P$, which is always unique if exists; see the end of Subsection 5.6 for details and related computability questions.

The result that we actually prove is somewhat more general than the theorem stated above in the following sense. Firstly, together with the Newton polyhedron, we describe the leading coefficients of the discriminant (i.e. the coefficients of monomials in the boundary of the Newton polyhedron) in terms of leading coefficients of the polynomials F_0, \dots, F_l (see Theorem 3.3 for $l = k$, Proposition 4.11 for $l = 0$, and Theorem 5.4 that reduces the general case to $l = 0$). Secondly, we do not assume that $A_0 = \dots = A_l$. Thirdly, we prefer a slightly more general context throughout the paper (see [G] for motivation): instead of polynomials on $(\mathbb{C} \setminus 0)^n$, we consider analytic functions on an arbitrary affine toric variety. Nevertheless, all our results and proofs can be translated back into the global setting, word by word, substituting germs of analytic functions on affine toric varieties with Laurent polynomials on complex tori, unbounded polyhedra with bounded ones, and the local version of the elimination theorem 3.3 with its global version [EKH]. For instance, the theorem stated above is exactly the global version of Theorem 5.10 with $A_0 = \dots = A_l$. We illustrate it with an example in Subsection 5.6.

Denote the Newton polyhedron of the discriminant of $F_0 = \dots = F_l = 0$ by $\mathcal{N}(\Delta_0, \dots, \Delta_l)$ for generic equations F_0, \dots, F_l with given Newton polyhedra $\Delta_0, \dots, \Delta_l$. Theorem 5.10 leads to a certain relation between Newton polyhedra of discriminants (higher additivity, see Subsection 5.4 for details), provided that the convex hulls of A_0, \dots, A_l are large enough and analogous (i.e. have the same number of faces as the convex hull of their sum $A_0 + \dots + A_l$):

$$\begin{aligned} \mathcal{N}(\Delta_0 + \Delta_1, \Delta_2, \dots, \Delta_l) &= \sum_{\mu=1}^{\infty} \mathcal{N}(\underbrace{\Delta_0, \dots, \Delta_0}_{\mu}, \underbrace{\Delta_1, \dots, \Delta_1}_{\mu-1}, \Delta_2, \dots, \Delta_l) + \\ &+ \mathcal{N}(\underbrace{\Delta_0, \dots, \Delta_0}_{\mu-1}, \underbrace{\Delta_1, \dots, \Delta_1}_{\mu}, \Delta_2, \dots, \Delta_l) + 2\mathcal{N}(\underbrace{\Delta_0, \dots, \Delta_0}_{\mu}, \underbrace{\Delta_1, \dots, \Delta_1}_{\mu}, \Delta_2, \dots, \Delta_l) \end{aligned}$$

(all but finitely many polyhedra in this sum are equal to $\{0\}$; in particular, for $l = k + 1$, this is the conventional additivity). Unexpectedly, the assumption that A_0, \dots, A_l are analogous can be significantly relaxed in some cases (see Subsection

5.4 for examples), and it would be interesting to know, to what extent it can be relaxed in general.

The paper is organized as follows. In Section 1, we recall necessary facts and notation, related to convex geometry and Newton polyhedra. In section 2, we study the universal case of our problem, which generalizes results of [GKZ] and [S94]. In Section 3, we study the special case $l = k$ of our problem (Theorem 3.3), which is a local version of elimination theory in the context of Newton polyhedra [EKh], and is based on a certain local version of D. Bernstein's formula (Theorem 1.15). In Section 4, we apply elimination theory to study the special case $l = 0$ (Theorem 4.10). In Section 5, we reduce the general case $0 \leq l \leq k$ to the case $l = 0$ by means of a classical technique, known as Cayley trick, or Lagrange multipliers (Theorem 5.4).

In particular, if the Newton polyhedra of F_0, \dots, F_l are not too "thin", then the discriminant set Σ is a hypersurface (see Propositions 4.2 for $l = 0$ or 5.2 for arbitrary l), and its Newton polyhedron can be computed by Theorems 4.10 and 5.4. By "thin" we mean Newton polyhedra, such that the collection A_0, \dots, A_l (see above) is dual defect. One simple test for non-dual-defectiveness is provided by Propositions 2.14 (for $l = 0$) and 2.24.

We also study the same problem as formulated in the beginning, with another definition of the discriminant: we can define the discriminant set as the minimal set S in $(\mathbb{C} \setminus 0)^n$, such that the restriction of the projection $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k \rightarrow (\mathbb{C} \setminus 0)^n$ to the complete intersection $\{F_0 = \dots = F_l = 0\}$ is a fiber bundle outside of S . This version of the problem is outlined in Subsection 5.5. For example, if A is the set of integer points of a Delzant polyhedron in the assumptions of the theorem stated above, then the Newton polyhedron of the equation of S equals

$$\sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = k+1}} \Delta_0^{a_0} \cdot \dots \cdot \Delta_l^{a_l}$$

(see Corollary 5.18). The counterpart of dual-defectiveness for this problem seems to behave much simpler: see Proposition 2.29 and Conjecture 2.28.

All answers are formulated in terms of mixed fiber polyhedra. This notion is introduced in Section 1.2 and, in more detail, in Appendix (existence, uniqueness and monotonicity of mixed fiber bodies is proved and a formula for the support function of a mixed fiber body is given).

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1 Mixed volumes, mixed fiber polyhedra, and Euler obstructions.

In this section, we recall relevant facts and notation from convex geometry: mixed fiber polyhedra ([McM], [EKh]), Euler obstructions of polyhedra ([MT]), relative mixed volume ([E05], [E06], [E09]), and the corresponding relative version of the Kouchnirenko-Bernstein formula ([B], [Kh]). Subsections 1.4 and 1.6 contain generalizations of the relative Kouchnirenko-Bernstein formula that provide a simple proof for the Matsui-Takeuchi formula for Euler obstructions and for the Gelfand-Kapranov-Zelevinsky decomposition formula; we do not need these generalizations for other purposes. Material from other subsections is used in the proof of our main result.

1.1 Relative mixed volume.

CLASSICAL MIXED VOLUME. Recall the notion of the mixed volume of bounded polyhedra. The set \mathcal{M} of all bounded polyhedra in \mathbb{R}^m is a semigroup with respect to Minkowski summation $P + Q = \{p + q \mid p \in P, q \in Q\}$.

DEFINITION 1.1. *The mixed volume of polyhedra is the symmetric multilinear function $MV : \underbrace{\mathcal{M} \times \dots \times \mathcal{M}}_m \rightarrow \mathbb{R}$, such that $MV(P, \dots, P)$ equals the volume of P for every $P \in \mathcal{M}$.*

LEMMA 1.2 ([Kh]). $MV(\Delta_1, \dots, \Delta_m) = 0$ if and only if $\dim \Delta_{i_1} + \dots + \Delta_{i_J} < J$ for some $i_1 < \dots < i_J$.

This fact is mentioned as obvious in [Kh], but we prefer to give a proof for the sake of completeness.

PROOF. (\Leftarrow) follows by an explicit computation if $\Delta_{i_1} = \dots = \Delta_{i_J} = C$ is a cube, and all other Δ_i are equal to an m -dimensional cube with a face C ; the general case can be reduced to this one by monotonicity of the mixed volume. In the other direction, consider points $a_i \in \Delta_i$ and $b_i \in \Delta_i$ such that the vectors $a_1 - b_1, \dots, a_m - b_m$ are in general position in the sense that the dimension of the space generated by $a_{i_1} - b_{i_1}, \dots, a_{i_J} - b_{i_J}$ is the maximal possible one for every subset $\{i_1, \dots, i_J\} \subset \{1, \dots, m\}$. By monotonicity of the mixed volume, the mixed volume of the segments, connecting a_i and b_i , equals zero, which means that the vectors $a_1 - b_1, \dots, a_m - b_m$ are linearly dependent. In particular, there exists a minimal subset $\{i_1, \dots, i_J\} \subset \{1, \dots, m\}$ such that the vectors $a_{i_1} - b_{i_1}, \dots, a_{i_J} - b_{i_J}$ are linearly dependent. They generate a proper subspace $L \subset \mathbb{R}^m$, and every $J - 1$ of them form a basis of L . If there exists $b'_{i_j} \in \Delta_{i_j}$ such that $a_{i_j} - b'_{i_j} \notin L$, then the vectors $a_{i_1} - b_{i_1}, \dots, a_{i_J} - b_{i_J}$ with $a_{i_j} - b'_{i_j}$ instead of $a_{i_j} - b_{i_j}$ generate a subspace $L' \not\supseteq L$, which contradicts the condition of general position. Thus, $\Delta_{i_1}, \dots, \Delta_{i_J}$ are contained in a $(J - 1)$ -dimensional subspace L , up to a parallel translation, and $\dim \Delta_{i_1} + \dots + \Delta_{i_J} < J$. \square

RELATIVE MIXED VOLUME. We need the following relative version of the mixed volume. For a convex polyhedral m -dimensional cone $\tau \subset (\mathbb{R}^m)^*$, denote its dual cone $\{x \in \mathbb{R}^m \mid \gamma(x) > 0 \text{ for } \gamma \in \tau\}$ by τ^\vee , and let \mathcal{M}_{τ^\vee} be the semigroup of all (unbounded) polyhedra of the form

$$\tau^\vee + \text{a bounded polyhedron}.$$

Consider the set $\mathcal{P}_{\tau^\vee} \subset \mathcal{M}_{\tau^\vee} \times \mathcal{M}_{\tau^\vee}$ of all ordered pairs of polyhedra (P, Q) , such that the symmetric difference $P \Delta Q$ is bounded. \mathcal{P}_{τ^\vee} is a semigroup with respect to Minkowski summation of pairs $(P, Q) + (C, D) = (P + C, Q + D)$.

DEFINITION 1.3 ([E05], [E06], [E09]). *The volume $V(P, Q)$ of a pair of polyhedra $(P, Q) \in \mathcal{P}_\Gamma$ is defined to be the difference $\text{Vol}(P \setminus Q) - \text{Vol}(Q \setminus P)$. The mixed volume of pairs of polyhedra is defined to be the symmetric multilinear function $MV : \underbrace{\mathcal{P}_{\tau^\vee} \times \dots \times \mathcal{P}_{\tau^\vee}}_m \rightarrow \mathbb{R}$, such that $MV((P, Q), \dots, (P, Q)) = V(P, Q)$ for every pair $(P, Q) \in \mathcal{P}_{\tau^\vee}$.*

Existence and uniqueness are proved in [E06] and [E09].

EXAMPLE 1.4. If $\tau^\vee = \{0\}$, then \mathcal{P}_{τ^\vee} is the set of pairs of convex bounded polyhedra, and the mixed volume of pairs $\text{MV}\left((P_1, Q_1), \dots, (P_m, Q_m)\right)$ equals the difference of classical mixed volumes of the collections P_1, \dots, P_m and Q_1, \dots, Q_m .

In general, the mixed volume of pairs can be expressed in terms of the classical mixed volume as follows.

LEMMA 1.5 ([E06], [E09]). *Let $\tilde{P}_i \subset P_i$ and $\tilde{Q}_i \subset Q_i$ be bounded polyhedra in \mathbb{R}^m , such that $\tilde{P}_i \setminus \tilde{Q}_i = P_i \setminus Q_i$ and $\tilde{Q}_i \setminus \tilde{P}_i = Q_i \setminus P_i$ for $i = 1, \dots, m$. Then*

$$\text{MV}\left((P_1, Q_1), \dots, (P_m, Q_m)\right) = \text{MV}(\tilde{P}_1, \dots, \tilde{P}_m) - \text{MV}(\tilde{Q}_1, \dots, \tilde{Q}_m).$$

Note that, for any pair (P_i, Q_i) , we can always find the requested bounded polyhedra \tilde{P}_i and \tilde{Q}_i .

The cone τ^\vee plays the role of the unit in the semigroup \mathcal{P}_{τ^\vee} :

LEMMA 1.6 ([E06], [E09]).

- 1) $\text{MV}\left((\tau^\vee, Q_1), (P_2, Q_2), \dots, (P_m, Q_m)\right) = \text{MV}\left((\tau^\vee, Q_1), (Q_2, Q_2), \dots, (Q_m, Q_m)\right),$
i.e. the left hand side does not depend on the choice of P_2, \dots, P_m ;
- 2) $\text{MV}\left((\tau^\vee, \tau^\vee), (P_2, Q_2), \dots, (P_m, Q_m)\right) = 0.$

MIXED VOLUME OF A PRISM. Let e_1, \dots, e_l be the standard basis of \mathbb{R}^l , and e_0 be $0 \in \mathbb{R}^l$. For bounded polyhedra P_0, \dots, P_l in \mathbb{R}^m and a subset $I \subset \{0, \dots, l\}$, denote the convex hull of the union of the polyhedra $P_i \times \{e_i\} \subset \mathbb{R}^m \oplus \mathbb{R}^l$, $i \in I$, by P_I . In what follows, it will be convenient to denote the mixed volume of bounded polyhedra Q_1, \dots, Q_m in \mathbb{R}^m by the monomial $Q_1 \cdot \dots \cdot Q_m$.

LEMMA 1.7.

$$\sum_{I \subset \{0, \dots, l\}} (-1)^{l+1-|I|} (m + |I| - 1)! \text{Vol}(P_I) = \sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = m}} m! P_0^{a_0} \cdot \dots \cdot P_l^{a_l}.$$

PROOF. Pick generic polynomials g_0, \dots, g_l whose Newton polyhedra are P_0, \dots, P_l . Compute the Euler characteristic of the hypersurface $\lambda_0 g_0 + \dots + \lambda_l g_l = 0$ in $\mathbb{CP}_{\lambda_0, \dots, \lambda_l}^l \times (\mathbb{C} \setminus 0)^m$ in the following two ways.

- 1) The subdivision of the toric variety $\mathbb{CP}_{\lambda_0, \dots, \lambda_l}^l \times (\mathbb{C} \setminus 0)^m$ into complex tori $T_I = \{\lambda_i = 0 \text{ for } i \notin I\}$ induces the subdivision of the hypersurface into pieces, whose Euler characteristics equal $(-1)^{l+1-|I|} (m + |I| - 1)! \text{Vol}(P_I)$ by the Kouchnirenko-Khovanskii formula [Kh]. By additivity, the Euler characteristic of the hypersurface $\lambda_0 g_0 + \dots + \lambda_l g_l = 0$ equals the left hand side of the desired equality.
- 2) Considering the projection of the hypersurface $\lambda_0 g_0 + \dots + \lambda_l g_l = 0$ to $(\mathbb{C} \setminus 0)^m$, we note that the fiber of this projection over a point $y \in (\mathbb{C} \setminus 0)^m$ equals \mathbb{CP}^l or \mathbb{CP}^{l-1} depending on whether $y \in \{g_0 = \dots = g_l = 0\}$ or not. Thus, integrating the Euler characteristic over fibers of this projection, we conclude that the Euler characteristic of the hypersurface $\lambda_0 g_0 + \dots + \lambda_l g_l = 0$ equals the Euler characteristic of the complete intersection $\{g_0 = \dots = g_l = 0\}$. Computing the latter one by the Kouchnirenko-Bernstein-Khovanskii formula [Kh], we get the right hand side of the desired equality. \square

1.2 Mixed fiber polyhedra.

MINKOWSKI INTEGRAL ([BS]). Denote the projections of the direct sum $\mathbb{R}^n \oplus \mathbb{R}^k$ onto the summands by p and q respectively. Consider an integer polyhedron $\Delta \subset \mathbb{R}^n \oplus \mathbb{R}^k$, whose projection $q(\Delta)$ is bounded, and denote the affine span of $q(\Delta)$ by S . Choose the volume form dx on S such that the volume of the image of S under the projection $\mathbb{R}^k \rightarrow \mathbb{R}^k/\mathbb{Z}^k$ equals $(1 + \dim S)!$, and consider the set $I \subset \mathbb{R}^n \oplus \mathbb{R}^k$ of points of the form $\int_{q(\Delta)} s(x) dx$, where s runs over all continuous sections of the projection $q : \Delta \rightarrow q(\Delta)$.

DEFINITION 1.8. The *Minkowski integral*, or the *fiber polytope* $\int \Delta$ is the closure of $p(I)$.

MIXED MINKOWSKI INTEGRAL ([McM]). Let $\tau^\vee \subset \mathbb{R}^n \subset \mathbb{R}^n \oplus \mathbb{R}^k$ be a convex polyhedral cone that does not contain a line. *The mixed Minkowski integral, or the mixed fiber polyhedron*, is the symmetric multilinear polyhedral-valued function $\text{MP} : \underbrace{\mathcal{M}_{\tau^\vee} \times \dots \times \mathcal{M}_{\tau^\vee}}_{k+1} \rightarrow \mathcal{M}_{\tau^\vee}$, such that, for every polyhedron $\Delta \in \mathcal{M}_{\tau^\vee}$,

$$\text{MP}(\Delta, \dots, \Delta) = \begin{cases} \int_{\tau^\vee} \Delta & \text{if } \dim q(\Delta) = n \\ \tau^\vee & \text{otherwise} \end{cases}.$$

See [McM], [EKh], or Appendix for existence, uniqueness and properties of the mixed fiber polyhedron.

EXAMPLE 1.9. If Δ is the product of $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^k$, then $\int \Delta$ equals $\left((n+1)! \text{Vol } Q\right) \cdot P$. If $n = 1$ and $\tau^\vee = \{0\}$, then $\text{MP}(\Delta_0, \dots, \Delta_n)$ is a segment of length $(n+1)! \text{MV}(\Delta_0, \dots, \Delta_n)$.

We need the following formula for the support function of the mixed fiber polyhedron. Let $\Delta_0, \dots, \Delta_k$ be polyhedra in \mathcal{M}_{τ^\vee} , denote the set of positive real numbers by $\mathbb{R}_+ \subset \mathbb{R}$, and the product $\mathbb{R}_+ \times q(\Delta_i) \subset \mathbb{R} \oplus \mathbb{R}^k$ by B_i . For every linear function $l : \mathbb{R}^n \rightarrow \mathbb{R}$, denote the image of Δ_i under the map $(l, \text{id}) : \mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R} \oplus \mathbb{R}^k$ by $l\Delta_i$.

PROPOSITION 1.10 (see Appendix). *The minimal value of a linear function $l : \mathbb{R}^n \rightarrow \mathbb{R}$ on $\text{MP}(\Delta_0, \dots, \Delta_k)$ equals*

$$(k+1)! \text{MV}\left((l\Delta_0, B_0), \dots, (l\Delta_k, B_k)\right)$$

if $l \in \tau$ and equals $-\infty$ otherwise.

The proof is given in Appendix under an inessential assumption that the polyhedra are bounded.

We also need a little more flexible version of the notation above. For an l -dimensional vector space $L \subset \mathbb{R}^k$, consider the semigroup $\mathcal{M}_{\tau^\vee}(L)$ of all polyhedra of the form

$$Q + \tau^\vee \times \{x\} \subset \mathbb{R}^n \oplus \mathbb{R}^k,$$

where x is a point in \mathbb{R}^k and Q is a bounded polyhedron in $\mathbb{R}^n \oplus L$.

DEFINITION 1.11. *The mixed Minkowski integral, or the mixed fiber polyhedron, is the symmetric multilinear polyhedral-valued function*

$$\text{MP} : \underbrace{\mathcal{M}_{\tau^\vee}(L) \times \dots \times \mathcal{M}_{\tau^\vee}(L)}_{l+1} \rightarrow \mathcal{M}_{\tau^\vee}(0),$$

such that, for every polyhedron $\Delta \in \mathcal{M}_{\tau^\vee}(L)$,

$$\text{MP}(\Delta, \dots, \Delta) = \begin{cases} \int_{\tau^\vee} \Delta & \text{if } \dim q(\Delta) = l \\ \tau^\vee & \text{otherwise} \end{cases}.$$

Note that the value $\text{MP}(\Delta_0, \dots, \Delta_l)$ does not depend on the choice of the cone τ^\vee and the space L in the definition above. More precisely, the cone τ^\vee is uniquely defined by the arguments $\Delta_0, \dots, \Delta_l$. The space L is also uniquely defined by $\Delta_0, \dots, \Delta_l$, provided that the projection $q(\Delta_0 + \dots + \Delta_l)$ is l -dimensional; otherwise, $\text{MP}(\Delta_0, \dots, \Delta_l) = \tau^\vee$ independently of the choice of L .

MINKOWSKI INTEGRAL OF A PRISM. In what follows, it is convenient to denote the mixed fiber polyhedron $\text{MP}(\Delta_0, \dots, \Delta_l)$ by the monomial $\Delta_0 \cdot \dots \cdot \Delta_l$, as well as we do for the mixed volume (this agrees with Example 1.9). Let e_1, \dots, e_l be the standard basis of \mathbb{R}^l , and e_0 be $0 \in \mathbb{R}^l$. For polyhedra P_0, \dots, P_l in \mathcal{M}_{τ^\vee} and a subset $I \subset \{0, \dots, l\}$, denote the convex hull of the union of the polyhedra $P_i \times \{e_i\} \subset \mathbb{R}^n \oplus \mathbb{R}^k \oplus \mathbb{R}^l$, $i \in I$, by P_I .

LEMMA 1.12. *For polyhedra P_0, \dots, P_l in \mathcal{M}_{τ^\vee} ,*

$$\begin{aligned} 1) \quad P_{\{0, \dots, l\}}^{k+l+1} &= \sum_{\substack{a_0 \geq 0, \dots, a_l \geq 0 \\ a_0 + \dots + a_l = k+1}} P_0^{a_0} \cdot \dots \cdot P_l^{a_l}, \\ 2) \quad \sum_{I \subset \{0, \dots, l\}} (-1)^{l+1-|I|} P_I^{k+|I|} &= \sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = k+1}} P_0^{a_0} \cdot \dots \cdot P_l^{a_l}. \end{aligned}$$

PROOF. These two formulas are equivalent by the inclusion-exclusion formula, and we prove the second one. Substituting mixed fiber polyhedra with mixed volumes of pairs by Proposition 1.10, and then with classical mixed volumes by Lemma 1.5, it is enough to prove the same formula for mixed volumes of bounded polyhedra, which is the statement of Lemma 1.7. \square

1.3 Kouchnirenko-Bernstein formula.

The relative version of the classical mixed volume participates in a certain relative version of the classical Kouchnirenko-Bernstein formula. To formulate it, we need some notation related to toric varieties, Newton polyhedra and intersection numbers.

TORIC VARIETIES. For a rational fan Σ in $(\mathbb{R}^m)^*$, the corresponding toric variety is denoted by \mathbb{T}^Σ . For every codimension 1 orbit T of \mathbb{T}^Σ , the primitive generator of the corresponding 1-dimensional cone of Σ is denoted by $\gamma(T)$. We assume that the union of cones of Σ is a closed convex cone τ , and denote its dual by $\tau^\vee \subset \mathbb{R}^m$.

If I is a very ample line bundle on \mathbb{T}^Σ , and a meromorphic section s of the bundle I has no zeros and no poles in the maximal torus of \mathbb{T}^Σ , then there exists a unique polyhedron $\Delta \in \mathcal{M}_{\tau^\vee}$, such that the multiplicity of every codimension 1 orbit T of \mathbb{T}^Σ in the divisor of poles and zeroes of the section s equals the maximal value of the linear function $-\gamma(T) : \mathbb{R}^m \rightarrow \mathbb{R}$ on the polyhedron Δ . Since the pair (I, s) is uniquely determined by this polyhedron Δ , we denote the line bundle I by I_Δ and the section s by s_Δ .

NEWTON POLYHEDRA. The union of all precompact orbits of the toric variety \mathbb{T}^Σ (i.e. the orbits, corresponding to the cones of Σ in the interior of τ) is denoted by \mathbb{T}_{comp}^Σ and is called the *compact part of \mathbb{T}^Σ* (it is indeed a compact set).

If f is an arbitrary germ of a holomorphic section of I_Δ near the compact set \mathbb{T}_{comp}^Σ , then the function f/s_Δ can be represented as a power series $\sum_{a \in \Delta} c_a x^a$ for x in the maximal torus $(\mathbb{C} \setminus 0)^m$ of the toric variety \mathbb{T}^Σ . The convex hull of the set $\{a \mid c_a \neq 0\} + \tau^\vee$ is an integer polyhedron in \mathcal{M}_{τ^\vee} . It is called the *Newton polyhedron* of f and is denoted by Δ_f . For any bounded $\Gamma \subset \mathbb{R}^m$, the polynomial $\sum_{a \in \Gamma} c_a x^a$ on $(\mathbb{C} \setminus 0)^m$ is denoted by f^Γ . If a is contained in a bounded face of the Newton polyhedron Δ_f , then the coefficient c_a is called a *leading coefficient* of f . Every section has finitely many leading coefficients.

INTERSECTION NUMBERS. Let f_1, \dots, f_k be continuous sections of complex line bundles I_1, \dots, I_k on a k -dimensional complex algebraic variety V , such that the set $\{f_1 = \dots = f_k = 0\}$ is compact. Consider the Chern class $c_i \in H^2(V, \{f_i \neq 0\}; \mathbb{Z})$ of the bundle I_i , localized near the zero locus of its section f_i . Then the *intersection number* of the divisors of the sections f_1, \dots, f_k is defined as $c_1 \smile \dots \smile c_k \in H^{2k}(V, \cup_i \{f_i \neq 0\}; \mathbb{Z}) = \mathbb{Z}$ and is denoted by $m(f_1 \cdot \dots \cdot f_k \cdot V)$.

In other words, if we consider smooth (non-holomorphic) perturbations $\tilde{f}_1, \dots, \tilde{f}_k$ of the sections f_1, \dots, f_k , such that the system $\tilde{f}_1 = \dots = \tilde{f}_k = 0$ has finitely many regular solutions near the set $\{f_1 = \dots = f_k = 0\}$, then each of the solutions can be assigned a weight ± 1 , depending on orientation of the base $d\tilde{f}_1, \dots, d\tilde{f}_k$ at this point. The intersection number $m(f_1 \cdot \dots \cdot f_k \cdot V)$ by definition equals the sum of these weights.

RELATIVE KOUCHNIRENKO-BERNSTEIN FORMULA. Let $\Delta_1, \dots, \Delta_m$ be integer polyhedra in \mathcal{M}_{τ^\vee} , and let $\tilde{\Delta}_1, \dots, \tilde{\Delta}_m$ be the Newton polyhedra of germs of sections f_1, \dots, f_m of the line bundles $I_{\Delta_1}, \dots, I_{\Delta_m}$ on the toric variety \mathbb{T}^Σ . We compute the intersection number of the divisors of the sections f_1, \dots, f_m in terms of the polyhedra $\Delta_1, \dots, \Delta_m$ and $\tilde{\Delta}_1, \dots, \tilde{\Delta}_m$, provided that the leading coefficients of f_1, \dots, f_m are in general position.

DEFINITION 1.13. For every face Γ of the sum of polyhedra $\Delta_1, \dots, \Delta_m$ in \mathbb{R}^p , the maximal collection of faces $\Gamma_1 \subset \Delta_1, \dots, \Gamma_m \subset \Delta_m$, such that $\Gamma_1 + \dots + \Gamma_m = \Gamma$, is said to be *compatible*.

For bounded faces, the word “maximal” can be omitted in this definition.

DEFINITION 1.14. The leading coefficients of the sections f_1, \dots, f_m are said to be *in general position*, if, for every collection of bounded compatible

faces $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_m$ of the polyhedra $\tilde{\Delta}_1, \dots, \tilde{\Delta}_m$, the system of polynomial equations $f_1^{\tilde{\Gamma}_1} = \dots = f_m^{\tilde{\Gamma}_m} = 0$ has no roots in the maximal torus $(\mathbb{C} \setminus 0)^m$.

THEOREM 1.15 (Relative Kouchnirenko-Bernstein formula, [E05], [E06], [E09]). *Let $\Delta_1, \dots, \Delta_m$ be integer polyhedra in \mathcal{M}_{τ^\vee} , and let $\tilde{\Delta}_1, \dots, \tilde{\Delta}_m$ be the Newton polyhedra of sections f_1, \dots, f_m of the line bundles $I_{\Delta_1}, \dots, I_{\Delta_m}$, such that the difference $\Delta_i \setminus \tilde{\Delta}_i$ is bounded for every i . Then*

1) *The intersection number $m(f_1 \cdot \dots \cdot f_m \cdot \mathbb{T}^\tau)$ is greater or equal than the mixed volume $m! \text{MV}\left((\Delta_1, \tilde{\Delta}_1), \dots, (\Delta_m, \tilde{\Delta}_m)\right)$.*

2) *This inequality turns into an equality if and only if leading coefficients of the sections f_1, \dots, f_m are in general position in the sense of Definition 1.14.*

1.4 Kouchnirenko-Bernstein-Khovanskii formula.

In the assumptions of Theorem 1.15, suppose that the first line bundle \mathcal{I}_{Δ_1} is trivial, i.e. $\Delta_1 = \tau^\vee$. The relative version of the Kouchnirenko-Bernstein-Khovanskii formula computes the Euler characteristic of the Milnor fiber of the function f_1 on the complete intersection $f_2 = \dots = f_k = 0$ for $k \leq m$, in terms of the Newton polyhedra of the sections f_1, \dots, f_k . At the end of this subsection, we also explain how to drop the assumption on triviality of \mathcal{I}_{Δ_1} .

To define the Milnor fiber of s_1 , it is convenient to fix a family of neighborhoods for the compact part of the toric variety \mathbb{T}^Σ . For instance, choose an integer point a_i on every infinite edge of Δ_1 , and let B_ε be the set of all $x \in (\mathbb{C} \setminus 0)^m$ such that $\sum_i |x^{a_i}| \leq \varepsilon$. Then its closure in the toric variety \mathbb{T}^Σ is a neighborhood of the compact part \mathbb{T}_{comp}^Σ .

DEFINITION 1.16. The Milnor fiber of the function f_1 on the complete intersection $f_2 = \dots = f_k = 0$ is the manifold $\{f_1 - \delta = f_2 = \dots = f_k = 0\} \cap B_\varepsilon$ for $|\delta| \ll \varepsilon \ll 1$.

DEFINITION 1.17. The leading coefficients of f_1, \dots, f_k , are said to be in general position, if, for every collection of bounded compatible faces $\tilde{\Gamma}_1, \dots, \tilde{\Gamma}_k$ of the polyhedra $\tilde{\Delta}_1, \dots, \tilde{\Delta}_k$, the systems of polynomial equations $f_1^{\tilde{\Gamma}_1} = \dots = f_k^{\tilde{\Gamma}_k} = 0$ and $f_2^{\tilde{\Gamma}_2} = \dots = f_k^{\tilde{\Gamma}_k} = 0$ define regular varieties in the maximal torus $(\mathbb{C} \setminus 0)^m$.

We denote the mixed volume of pairs of polyhedra $(P_1, Q_1), \dots, (P_m, Q_m)$ in \mathbb{R}^m by the monomial $(P_1, Q_1) \cdot \dots \cdot (P_m, Q_m)$.

THEOREM 1.18. *In the above assumptions, the Euler characteristic of the Milnor fiber of f_1 on the complete intersection $\{f_2 = \dots = f_k = 0\}$ equals*

$$(-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} (\Delta_1, \tilde{\Delta}_1)^{a_1} \cdot \dots \cdot (\Delta_k, \tilde{\Delta}_k)^{a_k},$$

provided that the leading coefficients of f_1, \dots, f_k are in general position in the sense of Definition 1.17.

This is proved in [O] for a regular affine toric variety (based on the idea of [V]), and in [MT2] for an arbitrary affine toric variety (in a more up to date language). Both arguments can be easily applied to an arbitrary (not necessary affine) toric variety, and also provide a formula for the ζ -function of monodromy of the function f_1 . However, since we restrict our consideration to the Milnor number in this paper, we prefer to give a much simpler proof by reduction to the global Kouchnirenko-Bernstein-Khovanskii formula.

PROOF. If the leading coefficients are in general position, then topology of the Milnor fiber only depends on the Newton polyhedra of f_1, \dots, f_k , and we can assume without loss of generality that $f_i = s_{\Delta_i} \cdot \tilde{f}_i$, where $\tilde{f}_1, \dots, \tilde{f}_k$ are Laurent polynomials on $(\mathbb{C} \setminus 0)^m$ and satisfy the condition of general position of [Kh]. Denote the Newton polyhedra of the polynomials $\tilde{f}_1, \dots, \tilde{f}_k$ and $\tilde{f}_1 - \delta$ with $\delta \neq 0$ by D_1, \dots, D_k and \tilde{D}_1 .

By the global Kouchnirenko-Bernstein-Khovanskii formula [Kh], the Euler characteristics of $\{\tilde{f}_1 = \dots = \tilde{f}_k = 0\}$ and $\{\tilde{f}_1 - \delta = \tilde{f}_2 = \dots = \tilde{f}_k = 0\}$ equal $(-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} D_1^{a_1} \cdot \dots \cdot D_k^{a_k}$ and $(-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} \tilde{D}_1^{a_1} \cdot D_2^{a_2} \cdot \dots \cdot D_k^{a_k}$ respectively.

Since the boundary of B_ε subdivides the set $\{\tilde{f}_1 - \delta = \tilde{f}_2 = \dots = \tilde{f}_k = 0\}$ into two parts, homeomorphic to the set $\{\tilde{f}_1 = \dots = \tilde{f}_k = 0\}$ and the Milnor fiber of f_1 on $\{f_2 = \dots = f_n = 0\}$, the Euler characteristic of the latter equals

$$(-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} \tilde{D}_1^{a_1} \cdot D_2^{a_2} \cdot \dots \cdot D_k^{a_k} - (-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} D_1^{a_1} \cdot \dots \cdot D_k^{a_k}$$

by additivity of Euler characteristic. This difference is equal to

$$(-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} (\Delta_1, \tilde{\Delta}_1)^{a_1} \cdot \dots \cdot (\Delta_k, \tilde{\Delta}_k)^{a_k}$$

by Lemmas 1.5 and 1.6. \square

Summing up the Euler characteristics of the Milnor fibers of $f_1|_{\{f_2=\dots=f_k=0\}}$ over all non-compact toric subvarieties in \mathbb{T}^Σ , we have the following formula for the Euler characteristic of the closure of Milnor fiber.

The fact that \mathcal{I}_{Δ_i} is a line bundle on the toric variety \mathbb{T}^Σ implies that, for every cone $\sigma \in \Sigma$, all interior points of σ , being considered as linear functions on the polyhedron Δ_i , attain their minimum on the same face of Δ_i . Denote this face by A_i , the codimension of the cone σ by q , and pick a vector $a_i \in A_i$. Then the shifted pairs $\mathcal{A}_i = (A_i - a_i, (A_i \cap \tilde{\Delta}_i) - a_i)$ are contained in the same rational q -dimensional subspace of \mathbb{R}^m , and their q -dimensional mixed volumes make sense. Denote the number $(-1)^{q-k} q! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = q}} \mathcal{A}_1^{a_1} \cdot \dots \cdot \mathcal{A}_k^{a_k}$ by χ_σ .

COROLLARY 1.19. *In the above assumptions, the Euler characteristic of the closure of the Milnor fiber of f_1 on the complete intersection $\{f_2 = \dots = f_n = 0\}$ equals $\sum_{\sigma \in \Sigma} \chi_\sigma$, provided that the leading coefficients of s_1, \dots, s_k are in general position.*

We also formulate a more general version of this theorem, with no assumptions on the triviality of the first line bundle (we do not need this generalization in what follows; since the proof is similar to that of Corollary 1.19, we omit it). Let f_1, \dots, f_k be germs of holomorphic sections of arbitrary line bundles $\mathcal{I}_{\Delta_1}, \dots, \mathcal{I}_{\Delta_k}$ near the compact part of a toric variety \mathbb{T}^Σ , and pick holomorphic sections t_1, \dots, t_k of these bundles in the closure of the set B_ε for a small ε . Varieties $B_\varepsilon \cap \{f_1 - t_1 = \dots = f_k - t_k = 0\}$ are diffeomorphic to each other for almost all collections of small sections (t_1, \dots, t_k) . Such variety is called the Milnor fiber of the complete intersection $\{f_1 = \dots = f_k = 0\}$.

THEOREM 1.20. *In the above assumptions, the Euler characteristic of the Milnor fiber of the complete intersection $\{f_1 = \dots = f_k = 0\}$ equals*

$$(-1)^{m-k} m! \sum_{\substack{a_1 > 0, \dots, a_k > 0 \\ a_1 + \dots + a_k = m}} (\Delta_1, \tilde{\Delta}_1)^{a_1} \cdot \dots \cdot (\Delta_k, \tilde{\Delta}_k)^{a_k},$$

provided that the leading coefficients of f_1, \dots, f_k are in general position.

1.5 Euler obstructions of polyhedra.

EULER OBSTRUCTIONS OF VARIETIES. Let $\bigsqcup_{\alpha \in \Lambda} U_\alpha$ be a Whitney stratification of a complex algebraic variety U . Pick a point x_0 in a stratum $U_{\alpha'}$ and consider a germ of an analytic function $f : (U, U_{\alpha'}) \rightarrow (\mathbb{C}, 0)$ at this point. The Euler characteristic of the set $\{x \in U_\alpha \mid f(x) = \delta, |x - x_0| \leq \varepsilon\}$ takes the same value for almost all germs f and all positive numbers $\delta \ll \varepsilon \ll 1$. This value does not depend on the choice of $x_0 \in U_{\alpha'}$, and we denote the negative of this value by $\mu^{\alpha', \alpha}$ (note that it equals 0 unless U_α is adjacent to $U_{\alpha'}$). We also set $\mu^{\alpha, \alpha} = 1$ for every $\alpha \in \Lambda$ and denote the $|\Lambda| \times |\Lambda|$ matrix with entries $\mu^{\alpha', \alpha}$ by M .

DEFINITION 1.21. The (α', α) -entry of the inverse matrix M^{-1} is denoted by $\epsilon^{\alpha', \alpha}$ and is called the *Euler obstruction* of the closure of the stratum U_α at a point of the stratum $U_{\alpha'}$.

REMARK. Adjacency of strata induces a partial order structure on the set Λ . If we consider the collections $\mu^{\alpha', \alpha}$ and $\epsilon^{\alpha', \alpha}$ as functions on the poset $\Lambda \times \Lambda$, these functions are called *Möbius inverse* (see e.g. [H] for details).

EXAMPLE 1.22. Let $(U, 0)$ be an m -dimensional isolated toric singularity (it consists of two strata $U \setminus \{0\}$ and $\{0\}$). Denote the m -dimensional cone of its fan by τ , the convex hull of $\tau^\vee \cap \mathbb{Z}^n \setminus \{0\}$ by H , and the volume of $\tau^\vee \setminus H$ by V . Then, by Corollary 1.19, the above matrix M equals $\begin{pmatrix} 1 & m!V - 2 \\ 0 & 1 \end{pmatrix}$ for even m and $\begin{pmatrix} 1 & -m!V \\ 0 & 1 \end{pmatrix}$ for odd m . Thus, the Euler obstruction of U equals $2 - m!V$ for even m and $m!V$ for odd m .

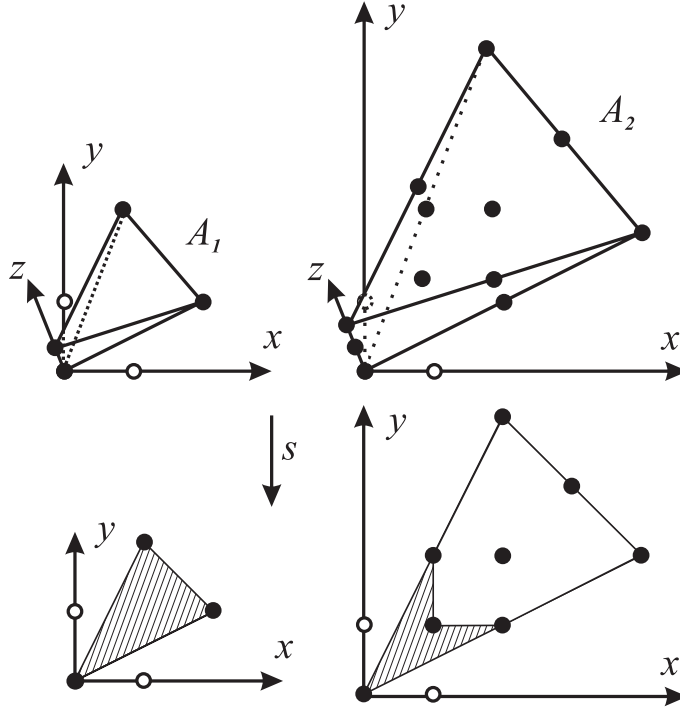
COMBINATORIAL COEFFICIENTS $c^{A', A}$ AND $e^{A', A}$.

DEFINITION 1.23. A subset $A' \subset A \subset \mathbb{Z}^k$ is called a *face* of A , if it can be represented as the intersection of A with a face of the convex hull of A .

For a face A' of a finite set $A \subset \mathbb{R}^k$, let M' and $M \subset \mathbb{R}^k$ be the vector spaces, parallel to the affine spans of A' and A respectively. We denote the projection $\mathbb{R}^k \rightarrow \mathbb{R}^k/M'$ by s , and choose the volume form η on M/M' such that the volume of $M/(M' + \mathbb{Z}^k)$ equals $(\dim M/M')!$. Then the η -volume of the difference of the convex hulls of $s(A)$ and $s(A \setminus A')$ is denoted by $c^{A',A} \in \mathbb{Z}$. Set $c^{A,A}$ to 1 and $c^{A',A}$ to 0 if A' is not a face of A .

EXAMPLE 1.24. The above difference is shown by hatching below for the following two examples:

- 1) $A_1 = \{(0, 0, 0), (0, 0, 1), (2, 1, 0), (1, 2, 1)\}$ and A'_1 is its vertical edge.
- 2) A_2 is the set of integer lattice points in the convex hull of $2 \cdot A_1$, and A'_2 is its vertical edge.



Consider the square matrix C with entries $c^{A'',A'}$, where A'' and A' run over the set of all faces of A , and define $e^{A'',A'}$ as the (A'', A') -entry of the inverse of C . Note that C is upper triangular with 1's on the diagonal, if we order faces of A by their dimension; in particular, the determinant of C equals 1, and its inverse is integer (although positivity of the entries of C does not imply positivity of the entries of its inverse).

DEFINITION 1.25. The number $e^{A',A}$ is called the *Euler obstruction of the set A at its face A'* .

For instance, restricting our attention to the four faces of the set A_2 adjacent to the vertical edge A'_2 in the previous example (including A'_2 itself), we obtain the Euler obstruction e^{A_2, A'_2} as the top right element in the matrix C^{-1} :

$$C = \begin{pmatrix} 1 & 1 & 1 & 2 \\ & 1 & 0 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & -1 & -1 & 0 \\ & 1 & 0 & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix}.$$

For an integer polyhedron $P \subset \mathbb{R}^m$ and its face Q , we call the numbers $c^{P \cap \mathbb{Z}^m, Q \cap \mathbb{Z}^m}$ and $e^{P \cap \mathbb{Z}^m, Q \cap \mathbb{Z}^m}$ the *Milnor number* and the *Euler obstruction* of the polyhedron P at its face Q , and denote them by $c^{Q,P}$ and $e^{Q,P}$.

EXAMPLE 1.26. For an integer polygon P at its vertex Q , denoting the convex hull of $\mathbb{Z}^2 \cap P \setminus \{Q\}$ by P_Q , we have

$$c^{Q,P} = 2 \cdot \text{area of } P \setminus P_Q, \quad e^{Q,P} = 2 - 2 \cdot \text{area of } P \setminus P_Q.$$

In particular, if $c^{Q,P} = e^{Q,P} = 1$, then the adjacent edges of the vertex Q can be brought to the lines

$$y = 0 \text{ and } x = 0$$

by a suitable affine change of coordinates in \mathbb{R}^2 that preserves the integer lattice. If $c^{Q,P} = 2$ (and $e^{Q,P} = 0$), then the adjacent edges of the vertex Q can be brought to the lines of the form

$$y = 0 \text{ and } y = \left(1 + \frac{1}{n}\right)x, \text{ where } n \in \mathbb{N}.$$

More generally, if $c^{Q,P} = s$, then the adjacent edges of the vertex Q can be brought to the lines of the form

$$y = 0 \text{ and } y = \left(p_0 + \frac{1}{p_1 + \cdots + \frac{1}{p_{2k}}}\right)x, \text{ where } p_i \in \mathbb{N} \text{ and } \sum_i p_{2i} = s$$

(the number in the brackets is the continued fraction of the sequence p_0, \dots, p_{2k}). It would be interesting to extend this to classification of r -dimensional rational convex polyhedral cones P such that $c^{0,P} = s$ or $e^{0,P} = s$ for small r and $|s|$; this problem is related to problems of classification of multidimensional continued fractions and sails (see e.g. [Kar]).

GEOMETRIC MEANING OF $c^{A',A}$ AND $e^{A',A}$. For a set $A = \{a_1, \dots, a_N\} \subset \mathbb{Z}^k$, such that the differences $a_i - a_j$ generate the lattice \mathbb{Z}^k , the closure of the image of the torus $(\mathbb{C} \setminus 0)^k$ under the inclusion $j : (\mathbb{C} \setminus 0)^k \rightarrow \mathbb{CP}^A$, $j(t) = (t^{a_1}, \dots, t^{a_N})$, is a toric variety, whose orbits are in one to one correspondence with faces of A . Its subdivision into the orbits is a Whitney stratification $\bigsqcup U_{A'}$, where A' runs over all faces of A . This stratification gives rise to the numbers $\mu^{A'',A'}$ and $\epsilon^{A'',A'}$ for every pair of faces A' and A'' , as defined in the beginning of this subsection.

Since, by Theorem 1.18, we have $\mu^{A'',A'} = (-1)^{\dim A'' - \dim A'} c^{A'',A'}$, then $\epsilon^{A'',A'} = (-1)^{\dim A'' - \dim A'} e^{A'',A'}$, which proves

PROPOSITION 1.27 ([MT]). *The Euler obstruction of the set $A \subset \mathbb{Z}^k$ at its face A' equals $(-1)^{\dim A' - \dim A}$ times the Euler obstruction of the toric variety, corresponding to A , at a point of its orbit, corresponding to A' .*

1.6 Multiplicities of non-degenerate complete intersections.

Here we formulate a corollary of the relative Kouchnerenko-Bernstein formula that leads to a simple proof the Gelfand-Kapranov-Zelevinsky decomposition formula (we do not need it for other purposes).

VARIETIES WITH MULTIPLICITIES. Let f_1, \dots, f_l be germs of holomorphic sections of complex line bundles I_1, \dots, I_l on a germ of a k -dimensional complex algebraic variety (V, x) , such that the set $\{f_1 = \dots = f_l = 0\}$ is smooth and $(k-l)$ -dimensional. Then we can choose germs of holomorphic functions f_{l+1}, \dots, f_k on (V, x) , such that the differentials of their restrictions to $\{f_1 = \dots = f_l = 0\}$ are linearly independent. The local topological degree of the map $(f_1, \dots, f_k) : (V, x) \rightarrow (\mathbb{C}^k, 0)$ does not depend on the choice of the germs f_{l+1}, \dots, f_k and is called *the multiplicity of the (local) complete intersection $f_1 = \dots = f_l = 0$ at its point x* .

If S_i , $i = 1, \dots, I$, are the irreducible components of a complete intersection $\{f_1 = \dots = f_l = 0\}$, and a_i is the multiplicity of this complete intersection at a smooth point of S_i , then we denote the cycle $\sum_{i=1}^I a_i S_i$ (i.e. a formal sum of irreducible varieties) by $[f_1 = \dots = f_l = 0]$.

If $S = \sum a_i S_i$ is a cycle in V (every S_i is an irreducible variety in V , and every a_i is a positive number), and $f : V \rightarrow W$ is a proper map, then we define the *image* $f_*(S)$ as follows. For every component S_i , denote the topological degree of the map $f : S_i \rightarrow f(S_i)$ by d_i , provided that $\dim f(S_i) = \dim S_i$ (otherwise, set $d_i = 0$ by definition). Then the *image* $f_*(S)$ is defined to be the sum $\sum a_i d_i f(S_i)$.

MULTIPLICITIES OF NONDEGENERATE COMPLETE INTERSECTIONS. Let $\Delta \subset \mathbb{R}^m$ be an m -dimensional integer polyhedron, let Σ be its dual fan, and let $\tilde{\Delta}$ be the Newton polyhedron of holomorphic sections f_1, \dots, f_l of the line bundle I_Δ on the toric variety \mathbb{T}^Σ . Denote the orbit of \mathbb{T}^Σ , corresponding to a codimension l face Γ of Δ , by \mathbb{T}^Γ . Then the multiplicity of the complete intersection $f_1 = \dots = f_l = 0$ at a generic point of the orbit \mathbb{T}^Σ can be computed as follows.

Let $p : \mathbb{R}^m \rightarrow \mathbb{R}^l$ be a projection such that $p(\mathbb{Z}^m) = \mathbb{Z}^l$ and $p(\Gamma)$ is a point.

LEMMA 1.28. *If $p(\tilde{\Delta})$ touches all faces of $p(\Delta)$ except the vertex $p(\Gamma)$, and the leading coefficients of f_1, \dots, f_l are in general position, then the closure of the orbit \mathbb{T}^Γ is a component of multiplicity $l! \text{Vol}(p(\Delta) \setminus p(\tilde{\Delta}))$ in the complete intersection $f_1 = \dots = f_l = 0$.*

The assumption that the Newton polyhedra of the sections f_1, \dots, f_l coincide is obviously redundant; it is introduced to simplify the notation.

PROOF. Let $\mathbb{T}_\Gamma \subset (\mathbb{C} \setminus 0)^m$ be the l -dimensional subtorus that acts trivially on the orbit \mathbb{T}^Γ . For every point $x \in \mathbb{T}^\Gamma$, consider the \mathbb{T}_Γ -invariant l -dimensional closed toric subvariety $H_x \subset \mathbb{T}^\Sigma$ that intersects the orbit \mathbb{T}^Γ at the point x .

We should prove that, for a generic point x , the intersection number \mathcal{J} of the variety H_x and the complete intersection $f_1 = \dots = f_l = 0$ at x makes sense and equals $l! \text{Vol}(p(\Delta) \setminus p(\tilde{\Delta}))$. To do so, we denote the restrictions of the line bundle I_Δ and its sections f_1, \dots, f_l to H_x by I' and f'_1, \dots, f'_l respectively and

note that $I' = I_{p(\Delta)}$ and the Newton polyhedra of f'_1, \dots, f'_l are equal to $p(\tilde{\Delta})$. On one hand, the intersection number of the divisors of f'_1, \dots, f'_l is equal to the desired intersection number \mathcal{J} , while on the other hand it makes sense and equals $l! \text{Vol}(p(\Delta) \setminus p(\tilde{\Delta}))$ by Theorem 1.15 for the sections f'_1, \dots, f'_l . \square

2 A -discriminants.

In this section, we discuss the universal case of our problem. For a collection of finite sets A_0, \dots, A_k in \mathbb{Z}^k , we recall the definition of the (A_0, \dots, A_k) -resultant (Subsection 2.1) and the A_0 -discriminant (Subsection 2.2). More generally, for every $l \leq k$, we introduce the so called (A_0, \dots, A_l) -discriminant (Subsection 2.4), and express it in terms of A -discriminants by means of the Cayley trick (Subsection 2.5).

The collection A_0, \dots, A_l is called dual defective, if the (A_0, \dots, A_l) -discriminant set is not a hypersurface, and we give a number of examples of sufficient conditions for non-dual defectiveness of a collection (Subsection 2.3 for $l = 0$ and Proposition 2.24 for arbitrary l). We also consider an alternative version of the (A_0, \dots, A_l) -discriminant set (the bifurcation set, see Definition 2.26), which is presumably always a hypersurface (see Proposition 2.29 and the subsequent conjecture). The technical proof of Propositions 2.24 and 2.29 and the alternative prove of the Gelfand-Kapranov-Zelevinsky decomposition formula are postponed till the end of this section.

2.1 Resultants and A -determinants ([S94] and [GKZ]).

RESULTANT. For a finite set $A \subset \mathbb{Z}^k$, denote the set of all Laurent polynomials of the form $\sum_{a \in A} c_a x^a$ on the complex torus $(\mathbb{C} \setminus 0)^k$ by $\mathbb{C}[A]$. Consider finite sets A_0, \dots, A_l in \mathbb{Z}^k , such that the dimension of the convex hull of $A_0 + \dots + A_l$ is not greater than l . Let Σ be the set of all collections $(\varphi_0, \dots, \varphi_l) \in \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l]$ such that $\varphi_0(y) = \dots = \varphi_l(y) = 0$ for some $y \in (\mathbb{C} \setminus 0)^k$.

If the closure of Σ is a hypersurface in $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l]$, then

- 1) it is defined by the equation $G = 0$ for a certain irreducible polynomial G of positive degree.
- 2) For a generic collection $(\varphi_0, \dots, \varphi_l) \in \Sigma$, the set $\{y \mid \varphi_0(y) = \dots = \varphi_l(y) = 0\}$ can be represented as $J \cdot T$, where

$$T = \{z \mid z^a = z^b \text{ for all } a \text{ and } b \text{ in the affine span of } A_0 + \dots + A_l\}$$

is a subtorus in $(\mathbb{C} \setminus 0)^k$, and $J \subset (\mathbb{C} \setminus 0)^k / T$ is a certain finite set, whose cardinality does not depend on the choice of $(\varphi_0, \dots, \varphi_l)$; we denote this cardinality by $d(A_0, \dots, A_l)$.

DEFINITION 2.1. If the closure of Σ is a hypersurface, then the polynomial $G^{d(A_0, \dots, A_l)}$ is called the (A_0, \dots, A_l) -resultant and is denoted by R_{A_0, \dots, A_l} , otherwise we set $R_{A_0, \dots, A_l} = 1$ by definition.

REMARK. The resultant is uniquely defined up to multiplication by a non-zero constant, and all equalities involving resultants should be understood correspondingly.

This definition differs from [S94] and [GKZ] by the exponent $d(A_0, \dots, A_l)$. This exponent slightly simplifies computations and can be easily expressed in terms of A as follows.

MULTIPLICITY OF THE RESULTANT. Let A_0, \dots, A_k be finite sets in \mathbb{Z}^k . The resultant R_{A_0, \dots, A_k} is by definition a certain power $d(A_0, \dots, A_k)$ of an irreducible polynomial. Here we recall an explicit formula for the number $d(A_0, \dots, A_k)$ and a criterion for triviality of the resultant R_{A_0, \dots, A_k} in terms of the sets A_0, \dots, A_k .

DEFINITION 2.2. The dimension of the convex hull of a finite set $A \subset \mathbb{R}^k$ is called the dimension of A and is denoted by $\dim A$.

DEFINITION 2.3. For every non-empty subset $J \subset \{0, \dots, k\}$, the difference $\dim \sum_{j \in J} A_j - |J|$ is called the *codimension of the collection* $A_j, j \in J$, and is denoted by $\text{codim } J$.

PROPOSITION 2.4 ([S94]).

- 1) There exists J with $\text{codim } J < -1$ if and only if $R_{A_0, \dots, A_k} = 1$.
- 2) If $\text{codim } J \geq -1$ for every J , then there exists the minimal set $J_0 \subset \{0, \dots, k\}$, such that $\text{codim } J_0 = -1$.

Under the assumption of Proposition 2.4(2), let $L_{\mathbb{Z}} \subset \mathbb{Z}^{k+1}$ be the lattice generated by the set $(\sum_{j \in J_0} A_j) \times \{1\} \subset \mathbb{Z}^k \oplus \mathbb{Z}$, and let $L \subset \mathbb{R}^{k+1}$ be the linear span of $L_{\mathbb{Z}}$, denote the number $|(L \cap \mathbb{Z}^{k+1})/L_{\mathbb{Z}}|$ by d_1 .

Denote the projection $\mathbb{R}^{k+1} \rightarrow \mathbb{R}^{k+1}/L$ by s , and choose the volume form η on \mathbb{R}^{k+1}/L such that the volume of $\mathbb{R}^{k+1}/(L + \mathbb{Z}^{k+1})$ equals $(k - |J_0| + 1)!$. Denote the η -mixed volume of the convex hulls of the sets $s(A_j \times \{1\})$, $j \in \{0, \dots, k\} \setminus J_0$, by d_2 .

PROPOSITION 2.5 ([E07], [E09]). In the notation above, $d(A_0, \dots, A_k) = d_1 \cdot d_2$.

A-DETERMINANT. Let l be the dimension of the affine span L of a finite set $A \subset \mathbb{Z}^k$, let t_1, \dots, t_k be the standard coordinates on \mathbb{R}^k , and let y_1, \dots, y_k be the standard coordinates on $(\mathbb{C} \setminus 0)^k$. Choose numbers i_1, \dots, i_l such that the functions t_{i_1}, \dots, t_{i_l} form a system of coordinates on L (for example, if $\dim A = k$, then $\{i_1, \dots, i_l\} = \{1, \dots, k\}$).

DEFINITION 2.6. The *A-determinant* is the polynomial E_A on $\mathbb{C}[A]$, defined by the equality

$$E_A(\varphi) = R_{A, \dots, A} \left(\varphi, y_{i_1} \frac{\partial \varphi}{\partial y_{i_1}}, \dots, y_{i_l} \frac{\partial \varphi}{\partial y_{i_l}} \right).$$

REMARK. The A -determinant is uniquely defined up to multiplication by a non-zero constant, and does not depend on the choice of the collection i_1, \dots, i_l , because changing this collection results in multiplication of the vector $(\varphi, y_{i_1} \frac{\partial \varphi}{\partial y_{i_1}}, \dots, y_{i_l} \frac{\partial \varphi}{\partial y_{i_l}})$ by a square non-degenerate matrix (by definition, the resultant $R_{A, \dots, A}$ is invariant with respect to multiplication of its argument by a non-degenerate matrix).

2.2 A -discriminants ([GKZ]).

For a finite set $A \in \mathbb{Z}^k$, let $\Sigma_A \subset \mathbb{C}[A]$ be the set of all polynomials $\varphi \in \mathbb{C}[A]$ such that both φ and its differential $d\varphi$ vanish at some point $y \in (\mathbb{C} \setminus 0)^k$.

If the closure of Σ_A is a hypersurface, then

1) it is defined by the equation $G = 0$ for a certain irreducible polynomial G of positive degree.

2) For a generic $\varphi \in \Sigma_A$, its singular locus $\{y \mid \varphi(y) = d\varphi(y) = 0\}$ has the form $J \cdot T$, where $T = \{z \mid z^a = z^b \text{ for all } a \text{ and } b \text{ in the affine span of } A\}$ is a subtorus in $(\mathbb{C} \setminus 0)^k$, and $J \subset (\mathbb{C} \setminus 0)^k / T$ is a finite set, whose cardinality $|J|$ does not depend on φ .

DEFINITION 2.7. If the closure of Σ_A is a hypersurface, then the polynomial $G^{|J|}$ is called the A -discriminant and is denoted by D_A ; otherwise we set $D_A = 1$.

REMARK. The discriminant is uniquely defined up to multiplication by a non-zero constant, and all equalities involving discriminants should be understood correspondingly.

This definition differs from [GKZ] by the exponent $|J|$. This exponent slightly simplifies computations in what follows, and can be easily expressed in terms of A : let $L_{\mathbb{Z}} \subset \mathbb{Z}^{k+1}$ be the lattice generated by the set $A \times \{1\} \subset \mathbb{Z}^k \oplus \mathbb{Z}$, and let $L \subset \mathbb{R}^{k+1}$ be its linear span, then $|J| = |(L \cap \mathbb{Z}^{k+1}) / L_{\mathbb{Z}}|$.

LEMMA 2.8. For every $A \subset \mathbb{Z}^m$, the discriminant D_A is a power of an irreducible polynomial.

PROOF. The set of all pairs $(\varphi, y) \in \mathbb{C}[A] \times (\mathbb{C} \setminus 0)^m$, such that $\varphi(y) = d\varphi(y) = 0$, is the total space of a vector bundle \mathcal{V} with the base space $(\mathbb{C} \setminus 0)^m$ and the projection $\mathbb{C}[A] \times (\mathbb{C} \setminus 0)^m \rightarrow (\mathbb{C} \setminus 0)^m$. Thus it is irreducible, thus its image Σ_A under the projection to $\mathbb{C}[A]$ is also irreducible. \square

Recall that the dimension of the convex hull of $A \subset \mathbb{Z}^k$ is denoted by $\dim A$.

LEMMA 2.9. For a generic $\varphi \in \mathbb{C}[A]$, the codimension of the set of all points $y \in (\mathbb{C} \setminus 0)^k$, such that $\varphi(y) = d\varphi(y) = 0$, equals $\dim A - \text{codim } \Sigma_A + 1$.

PROOF. The fiber of the vector bundle \mathcal{V} , introduced in the previous proof, has codimension $1 + \dim A$ in $\mathbb{C}[A]$. \square

A -discriminants and A -determinants are related as follows. For a polynomial $\varphi(y) = \sum_{a \in A} \varphi_a y^a$ and a face A' of the set A , denote the polynomial $\sum_{a \in A'} \varphi_a y^a$ by $\varphi^{A'}$. Recall that coefficients $c^{A', A}$ are introduced in Subsection 1.5.

PROPOSITION **2.10** ([GKZ]).

$$E_A(\varphi) = \prod_{A'} \left(D_{A'}(\varphi^{A'}) \right)^{c^{A', A}},$$

where A' runs over all faces of A , including $A' = A$.

This also follows from Theorem 1.15, see Subsection 2.8.

2.3 Dual defectiveness.

DEFINITION **2.11**. A finite set $A \subset \mathbb{Z}^k$ is said to be *dual defect*, if $D_A = 1$.

We recall a few simple facts about dual defect sets. There is also a well known way to decide combinatorially if the set is dual defect or not (Corollary 4.13), which will follow from our results on Newton polyhedra of discriminants. One more prospective combinatorial criterion for dual defectiveness is given by Conjecture 2.20 below. Note that these facts do not provide classification of dual defect sets, which is a much more complicated problem, and is solved only for Delzant polytopes by now, see [D]. An obvious but useful reformulation of this definition is as follows.

LEMMA **2.12**. *A finite set $A \subset \mathbb{Z}^k$, whose convex hull is k -dimensional, is dual defect if and only if a generic polynomial in $\mathbb{C}[A]$ has a singular point.*

PROOF. If a generic polynomial $\varphi \in \mathbb{C}[A]$ has a singular point, then a generic line of the form $\{\varphi - c \mid c \in \mathbb{C}\}$ intersects the set Σ_A , thus $\text{codim } \Sigma_A = 1$.

If $\text{codim } \Sigma_A = 1$ and the convex hull of A is k -dimensional, then a generic polynomial in Σ_A has an isolated singular point, thus all nearby polynomials in Σ_A has a singular point as well. \square

PROPOSITION **2.13** (Monotonicity; see also [CDS]). *If a subset A' of a finite set $A \subset \mathbb{Z}^k$ is not dual defect and is not contained in an affine hyperplane, then A is not dual defect.*

PROOF. We prove this by induction on $|A|$. Since dual defectiveness is by definition invariant with respect to parallel translations, the inductive step can be reduced to the following fact: if A' is not dual defect and is not contained in an affine hyperplane, then $A' \cup \{0\}$ is not dual defect. The latter implication follows by Lemma 2.12 \square

Dual defect sets are “thin” in many senses, for example:

PROPOSITION **2.14**. *If A is not contained in a union of two parallel hyperplanes, then it is not dual defect.*

This can be deduced from [CC]; we also give a simple self-contained proof, which consists of two simple properties of iterated circuits.

DEFINITION 2.15. A set $B \subset \mathbb{R}^m$ of cardinality $m + 2$ is called a *circuit*, if none of its cardinality $m + 1$ subsets is contained in an affine hyperplane.

A set $B \subset \mathbb{R}^m$ is called an *iterated circuit*, if it is not contained in an affine hyperplane, and if, after a suitable parallel translation, it can be represented as a disjoint union $\{0\} \sqcup B_1 \sqcup \dots \sqcup B_p$, such that the following condition is satisfied: denote the linear span of $\{0\} \sqcup B_1 \sqcup \dots \sqcup B_i$ by L_i for $i = 0, \dots, p$, then the projection $L_{i+1} \rightarrow L_{i+1}/L_i$ maps the union $\{0\} \sqcup B_{i+1}$ injectively onto a circuit in L_{i+1}/L_i .

The minimal possible number p in this representation is called the *depth* of the iterated circuit B .

EXAMPLE 2.16. Some iterated circuits do not contain circuits. The simplest example is the set of integer points in the unit ball centered at the origin in \mathbb{R}^m (this set is an iterated circuit: the sets B_i above are the pairs of its opposite points). If the cardinality of $B \subset \mathbb{R}^m$ is $2m$ or greater, then it is not an iterated circuit.

LEMMA 2.17. *An iterated circuit $B \subset \mathbb{Z}^m$ is not dual defect.*

PROOF. Proving this by induction on the depth of B , the inductive step can be reduced to the following statement by a suitable \mathbb{Q} -affine change of coordinates in \mathbb{R}^m : if $B \subset \mathbb{Z}^k$ is not dual defect and $B' \sqcup \{0\} \subset \mathbb{Z}^l$ is a circuit, then $B \sqcup B' \subset \mathbb{Z}^k \oplus \mathbb{Z}^l$ is not dual defect.

Since $B' \sqcup \{0\}$ is a circuit and B is not dual defect, then generic polynomials $\varphi \in \mathbb{C}[B]$ and $\psi \in \mathbb{C}[B']$ have singular points by Lemma 2.12, thus their sum $\varphi + \psi$, which is a generic polynomial in $\mathbb{C}[B \sqcup B']$, has a singular point as well, hence $B \sqcup B'$ is not dual defect by the same lemma. \square

LEMMA 2.18. *If a finite set $B \subset \mathbb{R}^m$ does not contain an iterated circuit, then it is contained in a union of two parallel hyperplanes.*

EXAMPLE 2.19. If B does not contain a circuit, it does not imply that B is contained in a union of two parallel hyperplanes. The simplest example is the same as the previous one.

PROOF. Without loss of generality we may assume that $0 \in B$, and pick a maximal subset $B' \subset B$, such that

- 1) $0 \in B'$, and
- 2) B' is an iterated circuit as a subset of its vector span L .

The image \tilde{B} of the set B under the projection along L consists of at most $k - \dim L + 1$ points, otherwise B were not maximal. Thus \tilde{B} is contained in the union of two parallel hyperplanes, and so does B . \square

PROOF OF PROPOSITION 2.14. If A is dual defect, then it does not contain an iterated circuit by Proposition 2.13 and Lemma 2.17, thus it is contained in the union of two parallel hyperplanes by Lemma 2.18. \square

In particular, we have proved the easy half of the following conjecture.

CONJECTURE 2.20. *A finite set $A \subset \mathbb{Z}^k$, whose convex hull is k -dimensional, is dual defect if and only if it does not contain an iterated circuit.*

Recall that D_A is a polynomial in the indeterminate coefficients φ_a of the polynomial $\varphi(y) = \sum_{a \in A} \varphi_a y^a$.

LEMMA 2.21. *For every non-dual defect $A \subset \mathbb{Z}^m$ and every $a \in A$, the discriminant D_A has positive degree as a polynomial of φ_a , unless it is a constant.*

PROOF. Since dual defectiveness is by definition invariant with respect to parallel translations, it is enough to prove the statement for $a = 0$, which follows from Lemma 2.12. \square

2.4 Discriminants D_{A_0, \dots, A_l} and B_{A_0, \dots, A_l} .

DISCRIMINANT D_{A_0, \dots, A_l} . Let A_0, \dots, A_l , $l \leq k$, be finite sets in \mathbb{Z}^k , and let $\Sigma_{A_0, \dots, A_l} \subset \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l]$ be the set of all collections of polynomials $(\varphi_0, \dots, \varphi_l)$, such that their differentials are linearly dependent at some point of the set $\{y \in (\mathbb{C} \setminus 0)^k \mid \varphi_0(y) = \dots = \varphi_l(y) = 0\}$. The union of the codimension 1 components of the closure $\overline{\Sigma_{A_0, \dots, A_l}}$ is defined by the equation $G = 0$ for a certain square-free polynomial G on $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l]$.

DEFINITION 2.22. The polynomial G is called the *reduced A -discriminant of degree (A_0, \dots, A_l)* and is denoted by $D_{A_0, \dots, A_l}^{red}$.

We assume that $l \leq k$, because the codimension of Σ_{A_0, \dots, A_l} is greater than 1 otherwise (however, one can still study the tropicalization of Σ_{A_0, \dots, A_l} instead of its Newton polyhedron, see [DFS] and [ST] for details).

If $l = k$, then $D_{A_0, \dots, A_k}^{red}$ is the reduced version of the resultant R_{A_0, \dots, A_k} (see Definition 2.1); if $l = 0$, then $D_{A_0}^{red}$ is the reduced version of the discriminant D_{A_0} (see Definition 2.7). In both of these cases, the discriminant set Σ_{A_0, \dots, A_l} is irreducible, and a combinatorial way to verify $\text{codim } \Sigma_{A_0, \dots, A_l} = 1$ is known (see Corollary 4.13(2) for $l = 0$ and Proposition 2.4(1) for $l = k$). In general, however, unlike in these two special cases, the set Σ_{A_0, \dots, A_l} may be not irreducible and not even of pure dimension. Thus, both its codimension 1 part $\{D_{A_0, \dots, A_l}^{red} = 0\}$ and its higher codimension part $\Sigma_{A_0, \dots, A_l} \setminus \{D_{A_0, \dots, A_l}^{red} = 0\}$ may be non-empty for $0 < l < k$ (see Example 2.25 below). Nevertheless, restricting our attention to the codimension 1 components, it turns out possible to express $D_{A_0, \dots, A_l}^{red}$ in terms of A -discriminants by means of the Cayley trick, See Theorem 2.31 below.

DUAL DEFECTIVENESS.

DEFINITION 2.23. A collection of sets A_0, \dots, A_l is said to be *dual defect*, if the closure of the set Σ_{A_0, \dots, A_l} is not a hypersurface.

One sufficient condition for non-dual-defectiveness is as follows:

PROPOSITION 2.24. *If neither of the sets A_0, \dots, A_l is contained in an affine hyperplane, and at least one of them is not dual defect, then the collection A_0, \dots, A_l is not dual defect.*

Proof is given in Subsection 2.6 below, as well as a more refined condition (Proposition 2.41), which is presumably a criterion. Note that neither of the conditions of this statement can be omitted in general, as the following examples demonstrate:

EXAMPLE 2.25. If $k = 2$ and $A_0 = A_1$ is the the set of vertices of the standard 2-dimensional simplex (which is dual defect), then Σ_{A_0, A_1} has codimension 2.

If $k = 2$, $A_0 = \{0, 1, 2\} \times \{0\}$ and $A_1 = \{0, 1\} \times \{0, 1\}$ (neither of these sets is dual defect, but $\dim A_0 < 2$), then $\overline{\Sigma_{A_0, A_1}}$ has two components. The first one consists of all pairs of polynomials of the form $(c(x-a)^2, b_{11}xy + b_{01}x + b_{10}y + b_{00})$, and has codimension 1. Another one consists of all pairs of polynomials of the form $(c_1(x-a_1)(x-a_2), c_2(x-a_1)(y-b))$, and has codimension 2.

DISCRIMINANT B_{A_0, \dots, A_l} . We also consider another possible definition of discriminant, such that the corresponding version of the non-dual defectiveness assumption is weaker than the conventional one. Let W be the set of all collections $(y, \varphi_0, \dots, \varphi_l) \in (\mathbb{C} \setminus 0)^k \times \mathbb{C}[A_0] \times \dots \times \mathbb{C}[A_l]$, such that $\varphi_0(y) = \dots = \varphi_l(y) = 0$. Let $S_{A_0, \dots, A_l} \subset \mathbb{C}[A_0] \times \dots \times \mathbb{C}[A_l]$ be the minimal (closed) set, such that the projection $W \rightarrow \mathbb{C}[A_0] \times \dots \times \mathbb{C}[A_l]$ is a fiber bundle outside of S .

DEFINITION 2.26. The set S_{A_0, \dots, A_l} is called the *bifurcation set*. The collection A_0, \dots, A_l is said to be *B-nondegenerate*, if S_{A_0, \dots, A_l} is a hypersurface. In this case the equation of S_{A_0, \dots, A_l} is denoted by B_{A_0, \dots, A_l} and is called the *bifurcation discriminant*.

In contrast to the discriminant $D_{A_0, \dots, A_l}^{red}$, the bifurcation discriminant takes “singularities of the system $\varphi_0 = \dots = \varphi_l = 0$ at infinity” into account.

EXAMPLE 2.27. In the notation of the previous example ($k = 2$, $A_0 = \{0, 1, 2\} \times \{0\}$, $A_1 = \{0, 1\} \times \{0, 1\}$), despite the discriminant set $\overline{\Sigma_{A_0, A_1}}$ is not of pure dimension, the bifurcation set $S_{A_0, A_1} \supset \overline{\Sigma_{A_0, A_1}}$ is a hypersurface that consists of five components. A generic point in each of these components is as follows (the codimension 2 component of the discriminant set is the intersection of the last two components):

$$\begin{aligned} & c(x-a)^2, b_{11}xy + b_{10}y + b_{01}x + b_{00}; \\ & a_1x^2 - a_2x, b_{11}xy + b_{10}y + b_{01}x + b_{00}; \\ & a_1x - a_2, b_{11}xy + b_{10}y + b_{01}x + b_{00}; \\ & c_1(x-a_1)(x-a_2), c_2(xy - a_1y + b_{01}x + b_{00}); \\ & c_1(x-a_1)(x-a_2), c_2(b_{11}xy + b_{10}y + x - a_1). \end{aligned}$$

In particular, we have

$$B_{A_0, A_1} = D_{A_0, A_1}^{red} \cdot D_{\{(0,0)\}, A_1}^{red} \cdot D_{\{(2,0)\}, A_1}^{red} \cdot D_{A_0, \{0,1\} \times \{1\}}^{red} \cdot D_{A_0, \{0,1\} \times \{0\}}^{red}.$$

These observations generalize as follows.

CONJECTURE 2.28. *All collections are B-nondegenerate.*

PROPOSITION 2.29. *If the convex hulls of finite sets A_0, \dots, A_l in \mathbb{Z}^k have the same dual fan, then the collection A_0, \dots, A_l is B-nondegenerate.*

In particular, if a collection consists of one set, then it is B-nondegenerate, in contrast to dual-defectiveness. The proof is given in Subsection 2.7.

LEMMA 2.30. *The bifurcation discriminant B_{A_0, \dots, A_l} equals the least common multiple of the discriminants $D_{A'_0, \dots, A'_l}$ over all compatible collections of faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$.*

We omit the proof, since it follows by definitions. In the next subsection, we explicitly decompose the discriminants $D_{A'_0, \dots, A'_l}$ into irreducible factors and compute the desired least common multiple (Corollary 2.32).

2.5 Cayley trick.

Let e_0, \dots, e_l be the standard basis in \mathbb{Z}^{l+1} . For $J \subset \{0, \dots, l\}$, denote the set $\bigcup_{j \in J} A_j \times \{e_j\}$ by $A_J \subset \mathbb{Z}^k \oplus \mathbb{Z}^{l+1}$. We identify the space $\bigoplus_{j \in J} \mathbb{C}[A_j]$ with the space $\mathbb{C}[A_J]$ by identifying a collection of polynomials $\varphi_j \in \mathbb{C}[A_j]$, $j \in J$, on the complex torus $(\mathbb{C} \setminus 0)^n$ with the polynomial $\sum_{j \in J} \lambda_j \varphi_j$ on $(\mathbb{C} \setminus 0)^k \times (\mathbb{C} \setminus 0)^{l+1}$, where $\lambda_0, \dots, \lambda_l$ are the standard coordinates on $(\mathbb{C} \setminus 0)^{l+1}$. This identification allows us to regard the discriminant $D_{A_J}^{\text{red}}$ as a polynomial on the space $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l]$.

THEOREM 2.31 (Cayley trick). *The discriminant $D_{A_0, \dots, A_l}^{\text{red}}$ equals the product of the discriminants $D_{A_J}^{\text{red}}$ over all subsets $J \subset \{0, \dots, l\}$, such that $\text{codim } J \leq \text{codim } J'$ for every $J' \supset J$ (recall that $\text{codim } J$ stands for the difference $\dim \sum_{j \in J} A_j - |J|$).*

For $l = k$, this is proved in [GKZ]; the only admissible J is $\{0, \dots, k\}$ in this case. If neither of A_i is contained in an affine hyperplane, then $\{0, \dots, k\}$ is the only admissible J as well. Lemma 2.30 leads to a similar formula for the bifurcation discriminant:

COROLLARY 2.32. *The bifurcation discriminant B_{A_0, \dots, A_l} equals the product of the discriminants $D_{A'_{\{j_1, \dots, j_p\}}}^{\text{red}}$ over all collections of compatible faces $A'_{j_1} \subset A_{j_1}, \dots, A'_{j_p} \subset A_{j_p}$ that can be extended to a collection of compatible faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$ such that $\dim \sum_{j \in J} A'_j - \dim \sum_i A'_{j_i} \geq |J| - p$ for every $J \supset \{j_1, \dots, j_p\}$.*

The proof of Theorem 2.31 is given at the end of this subsection and is based on the following construction.

DEFINITION 2.33. We define $\Sigma_{\{j_0, \dots, j_q\}}$ as the set of all collections $(\varphi_0, \dots, \varphi_l) \subset \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l]$, such that

- 1) $\varphi_0(x) = \dots = \varphi_l(x) = 0$ for some $x \in (\mathbb{C} \setminus 0)^k$, and
- 2) $\lambda_{j_0} d\varphi_{j_0}(x) + \dots + \lambda_{j_q} d\varphi_{j_q}(x) = 0$ for some $(\lambda_{j_0}, \dots, \lambda_{j_q}) \in (\mathbb{C} \setminus 0)^{q+1}$.

We have two options for each set Σ_J :

LEMMA 2.34. *A) If A_J is not dual defect, and $\text{codim } J \leq \text{codim } J'$ for every $J' \supset J$, then the closure of Σ_J is defined by the equation $D_{A_J}^{\text{red}} = 0$.
B) Otherwise, $\text{codim } \Sigma_J > 1$.*

The proof of this lemma is given below and is based on the following important fact:

LEMMA 2.35 ([Kh]). *Generic polynomials $\psi_i \in \mathbb{C}[A_i]$, $i = 0, \dots, l$, have a common root in $(\mathbb{C} \setminus 0)^n$ if and only if $\dim A_{i_1} + \dots + A_{i_q} \geq q$ for every sequence $0 \leq i_1 < \dots < i_q \leq l$.*

This fact is mentioned as obvious in [Kh], but we prefer to give a proof for the sake of completeness.

PROOF. If $l \geq k$ then the statement is obvious, because, on one hand, generic polynomials ψ_0, \dots, ψ_l have no common roots and, on the other hand, we have $\dim A_0 + \dots + A_l < l + 1$. The case $l < k - 1$ can be reduced to the case $l = k - 1$ by introducing arbitrary finite sets A_{l+1}, \dots, A_{k-1} , whose convex hulls are k -dimensional, and considering generic polynomials $\psi_i \in \mathbb{C}[A_i]$, $i = 0, \dots, k - 1$. Finally, if $l = k - 1$, then the number of common roots of generic polynomials $\psi_0, \dots, \psi_{k-1}$ equals $k!$ times the mixed volume of the convex hulls of the sets A_0, \dots, A_{k-1} , which is non-zero under the assumption of the lemma by Lemma 1.2. \square

PROOF OF LEMMA 2.34. First, note that the image of the set Σ_J under the natural projection $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l] \rightarrow \bigoplus_{j \in J} \mathbb{C}[A_j]$ is contained in Σ_{A_J} . In particular, if $\text{codim } \Sigma_{A_J} > 1$, then the set Σ_J satisfies both of the statements (A) and (B) of Lemma 2.34, independently of $\text{codim } J'$ for $J' \supset J$. Thus, we can assume that $\text{codim } \Sigma_{A_J} = 1$, i.e. A_J is not dual defect.

Consider the vector space $L \subset \mathbb{R}^n$, parallel to the affine span of the sum $\sum_{j \in J} A_j$, denote the projection $\mathbb{R}^k \rightarrow \mathbb{R}^k/L$ by p_L , and consider the torus $T_L = \{x \mid x^a = 1 \text{ for } a \in L\} \subset (\mathbb{C} \setminus 0)^k$. Then, for a generic polynomial in $\mathbb{C}[A_j]$, its restriction to T_L is a generic polynomial in $\mathbb{C}[p_L A_j]$. Thus, by Lemma 2.35, generic polynomials $\varphi_i \in \mathbb{C}[A_i]$, $i \notin J$, have a common zero on a torus $c \cdot T_L$, $c \in (\mathbb{C} \setminus 0)^k$, if and only if every subset $I \subset \{0, \dots, l\} \setminus J$ satisfies inequality $\dim \sum_{i \in I} p_L(A_i) \geq |I|$, i.e. $\text{codim } I \cup J \geq \text{codim } J$.

PROOF OF PART A. For $J = \{j_0, \dots, j_q\}$ and a polynomial $\lambda_{j_0} \varphi_{j_0} + \dots + \lambda_{j_q} \varphi_{j_q} \in \mathbb{C}[A_J]$, the set

$$\{x \in (\mathbb{C} \setminus 0)^k \mid \varphi_{j_0}(x) = \dots = \varphi_{j_q}(x) = 0 \text{ and } \lambda_{j_0} d\varphi_{j_0}(x) + \dots + \lambda_{j_q} d\varphi_{j_q}(x) = 0 \\ \text{for some } (\lambda_{j_0}, \dots, \lambda_{j_q}) \in (\mathbb{C} \setminus 0)^{q+1}\}$$

is non-empty and preserved under multiplication by elements of T_L , thus it contains a torus $c \cdot T_L$, $c \in (\mathbb{C} \setminus 0)^k$. Thus, generic polynomials $\varphi_i \in \mathbb{C}[A_i]$, $i \notin J$, have a common zero on it under the assumption of Part A. Thus, every fiber of the projection $\Sigma_J \rightarrow \Sigma_{A_J}$ is Zariski open in the corresponding fiber of the ambient projection $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l] \rightarrow \bigoplus_{j \in J} \mathbb{C}[A_j]$. Thus, $\text{codim } \Sigma_J = \text{codim } \Sigma_{A_J} = 1$.

PROOF OF PART B. For $J = \{j_0, \dots, j_q\}$ and a generic polynomial $\lambda_{j_0} \varphi_{j_0} + \dots + \lambda_{j_q} \varphi_{j_q}$ in $\mathbb{C}[A_J]$, the set

$$\{x \in (\mathbb{C} \setminus 0)^k \mid \varphi_{j_0}(x) = \dots = \varphi_{j_q}(x) = 0 \text{ and } \lambda_{j_0} d\varphi_{j_0}(x) + \dots + \lambda_{j_q} d\varphi_{j_q}(x) = 0 \\ \text{for some } (\lambda_{j_0}, \dots, \lambda_{j_q}) \in (\mathbb{C} \setminus 0)^{q+1}\}$$

consists of finitely many tori $c_s \cdot T_L$, $c_s \in (\mathbb{C} \setminus 0)^k$, $s = 1, \dots, S$, because A_J is not dual defect. Thus, generic polynomials $\varphi_i \in \mathbb{C}[A_i]$, $i \notin J$, have no common

zero on it under the assumption of Part B. Thus, a generic fiber of the projection $\Sigma_J \rightarrow \Sigma_{A_J}$ has codimension 1 or greater in the corresponding fiber of the ambient projection $\mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_l] \rightarrow \bigoplus_{j \in J} \mathbb{C}[A_j]$. Thus, $\text{codim } \Sigma_J \geq \text{codim } \Sigma_{A_J} > 1$. \square

PROOF OF THEOREM 2.31. The desired set Σ_{A_0, \dots, A_l} is the union of the sets Σ_J over all $J \subset \{0, \dots, l\}$. By Lemma 2.34, the product of the discriminants, mentioned in the formulation of the theorem, vanishes at the union of codimension 1 components of the closure $\overline{\Sigma_{A_0, \dots, A_l}}$. Since all non-constant discriminants in this product are irreducible and distinct by Lemmas 2.8 and 2.21, then this product is a square-free polynomial and thus equals $D_{A_0, \dots, A_l}^{\text{red}}$. \square

2.6 Proof of Proposition 2.24.

The proof relies upon Lemma 2.34 and relevant notation from the previous subsection. We split Proposition 2.24 into the following two lemmas.

LEMMA 2.36. *If A_0 is not dual defect, and $\dim A_i = k$ for every $i = 0, \dots, l$, then $A_{\{0, \dots, l\}}$ is not dual defect.*

LEMMA 2.37. *If $A_{\{0, \dots, l\}}$ is not dual defect, and $\dim A_i = k$ for every $i = 0, \dots, l$, then Σ_{A_0, \dots, A_l} is irreducible of codimension 1.*

Proof of both of these lemmas is given below and is based on the following fact.

LEMMA 2.38. *If $\dim A = k$, $\varphi \in \mathbb{C}[A]$, $\varphi(y_0) = 0$, and a vector $v \in \mathbb{C}^n$ is close enough to $d\varphi(y_0)$, then there exists $\tilde{\varphi} \in \mathbb{C}[A]$ near φ , such that $\tilde{\varphi}(y_0) = 0$ and $d\tilde{\varphi}(y_0) = v$.*

PROOF. One can readily verify this statement if A is of cardinality $k+1$ and $\varphi = 0$. Thus, if $A_0 \subset A$ is the set of vertices of a k -dimensional simplex, then there exists a small $\psi \in \mathbb{C}[A_0]$ such that $d\psi(y_0) = v - d\varphi(y_0)$, and we can set $\tilde{\varphi} = \varphi + \psi$. \square

PROOF OF LEMMA 2.36. Since dual defectiveness is preserved by parallel translations, we can assume without loss of generality that $0 \in A_0$. By Lemma 2.9, a generic polynomial $\varphi_0 \in \Sigma_{A_0}$ has an isolated singular point y_0 . By Lemma 2.38, generic polynomials $\varphi_i \in \mathbb{C}[A_i]$, such that $\varphi_i(y_0) = 0$, define a non-degenerate complete intersection $\varphi_1 = \dots = \varphi_l = 0$, passing through y_0 and transversal to the hypersurface $\varphi_0 = 0$ in a punctured neighborhood of y_0 . Then, for generic $\tilde{\varphi}_i \in \mathbb{C}[A_i]$ near φ_i , $i = 0, \dots, l$, there exists a small number ε such that the non-degenerate complete intersection $\tilde{\varphi}_1 = \dots = \tilde{\varphi}_l = 0$ is tangent to the smooth hypersurface $\tilde{\varphi}_0 = \varepsilon$ at some point near y_0 , which implies that $(\tilde{\varphi}_0 - \varepsilon, \tilde{\varphi}_1, \dots, \tilde{\varphi}_l) \in \Sigma_{A_0, \dots, A_l}$. Thus, $\text{codim } \Sigma_{A_0, \dots, A_l} = 1$ near the point $(\varphi_0, \dots, \varphi_l)$.

Since $\text{codim } \Sigma_{A_0, \dots, A_l} = 1$ at some point of this set,

$$\Sigma_{A_0, \dots, A_l} = \bigcup_{J \subset \{0, \dots, l\}} \Sigma_J,$$

$\text{codim } \Sigma_J > 1$ for $J \subsetneq \{0, \dots, l\}$ by Lemma 2.34(B),

$$\text{and } \Sigma_{\{0, \dots, l\}} = \Sigma_{A_{\{0, \dots, l\}}},$$

we have $\text{codim } \Sigma_{A_{\{0, \dots, l\}}} = 1$, thus $A_{\{0, \dots, l\}}$ is not dual defect. \square

PROOF OF LEMMA 2.37. For every collection $(\varphi_0, \dots, \varphi_l) \in \Sigma_{A_0, \dots, A_l}$, there exists a point y_0 , such that $\varphi_0(y_0) = \dots = \varphi_l(y_0) = 0$ and $d\varphi_0(y_0), \dots, d\varphi_l(y_0)$ are linearly dependent. There exist vectors v_0, \dots, v_l near $d\varphi_0(y_0), \dots, d\varphi_l(y_0)$, such that $\sum_{j=0}^l \lambda_j v_j = 0$ with $\lambda_j \neq 0$ for every $j = 0, \dots, l$. Thus, by Lemma 2.38, there exists a collection $(\tilde{\varphi}_0, \dots, \tilde{\varphi}_l)$ near $(\varphi_0, \dots, \varphi_l)$, such that

$$\sum_{j=0}^l \lambda_j d\tilde{\varphi}_j(y_0) = 0$$

with $\lambda_j \neq 0$ for every $j = 0, \dots, l$. This means that $(\lambda_0, \dots, \lambda_l, y_0) \in \Sigma_{A_{\{0, \dots, l\}}}$. Thus, Σ_{A_0, \dots, A_l} contains the irreducible codimension 1 set $\Sigma_{A_{\{0, \dots, l\}}}$ and is contained in its closure, which completes the proof. \square

REFINEMENT OF PROPOSITION 2.24. Consider finite sets A_0, \dots, A_l in \mathbb{Z}^k and a subset $J \subset \{0, \dots, l\}$.

DEFINITION 2.39. J is said to be *reliable*, if A_J is not dual defect, and $\text{codim } J \leq \text{codim } J'$ for every $J' \supset J$ (recall that $\text{codim } J$ stands for the difference $\dim \sum_{j \in J} A_j - |J|$).

Note that J is reliable if and only if Σ_J is a hypersurface.

DEFINITION 2.40. Vectors v_0, \dots, v_l are said to be J -linearly dependent, if $\sum_{j \in J} \lambda_j v_j = 0$ with $\lambda_j \neq 0$ for every $j \in J$.

For a set $A \subset \mathbb{R}^k$, denote the space of all linear functions, whose restrictions to A are constant, by $A^\perp \subset (\mathbb{R}^k)^*$. The following refined version of Proposition 2.24 is presumably a criterion for non-dual defectiveness of a collection of sets.

PROPOSITION 2.41. *A collection of finite sets A_0, \dots, A_l in \mathbb{Z}^k is not dual defect, if $\text{codim } J \geq -1$ for every $J \subset \{0, \dots, l\}$, and there exists a J -linearly dependent collection $\tilde{v}_0 \in A_0^\perp, \dots, \tilde{v}_l \in A_l^\perp$ with a reliable J in every neighborhood of every linearly dependent collection of vectors $v_0 \in A_0^\perp, \dots, v_l \in A_l^\perp$.*

We can formulate Lemma 2.38 with no assumption $\dim A = k$ as follows. Denote the natural identification $T_y(\mathbb{C} \setminus 0)^n \rightarrow T_{(1, \dots, 1)}(\mathbb{C} \setminus 0)^n$ by e_y .

LEMMA 2.42. *Suppose that $\varphi \in \mathbb{C}[A]$, $\varphi(y) = 0$, and $v \in \mathbb{C}^n$ is near $d\varphi(y)$. Then every neighborhood of φ contains $\tilde{\varphi} \in \mathbb{C}[A]$ with $\tilde{\varphi}(y) = 0$ and $d\tilde{\varphi}(y) = v$ if and only if $e_y(v) \in A^\perp$.*

PROOF OF PROPOSITION 2.41. For a collection $(\varphi_0, \dots, \varphi_l) \in \Sigma_{A_0, \dots, A_l}$, choose a point y , such that $\varphi_0(y) = \dots = \varphi_l(y) = 0$ and the differentials $d\varphi_0(y), \dots, d\varphi_l(y)$ are linearly dependent, and choose nearby vectors v_0, \dots, v_l to be J -linearly dependent for a reliable J . By Lemma 2.42, we have $\tilde{\varphi}_i(y) = 0$ and $d\tilde{\varphi}_i(y) = v_i$ for some $\tilde{\varphi}_i$ near φ_i , thus the collection $(\tilde{\varphi}_0, \dots, \tilde{\varphi}_l)$ is contained in the hypersurface Σ_J . \square

2.7 Proof of Proposition 2.29.

See Subsection 2.5 for the definition of A_J and Σ_J for $J \subset \{0, \dots, l\}$.

We prove the statement by induction on k . For a subset $A' \subset A$, there is a natural projection from $\mathbb{C}[A]$ to $\mathbb{C}[A']$ that assigns the polynomial $\sum_{a \in A'} c_a y^a$ to a polynomial $\sum_{a \in A} c_a y^a$. For subsets $A'_0 \subset A_0, \dots, A'_l \subset A_l$, we denote the preimage of the set $B \subset \mathbb{C}[A'_0] \oplus \dots \oplus \mathbb{C}[A'_l]$ under this projection by \tilde{B} .

The bifurcation set S_{A_0, \dots, A_l} contains the set $\tilde{S}_{A'_0, \dots, A'_l}$ for every collection of compatible faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$. The difference of S_{A_0, \dots, A_l} and $\bigcup_{A'_0, \dots, A'_l} \tilde{S}_{A'_0, \dots, A'_l}$ is contained in the set Σ_{A_0, \dots, A_l} , which is the union of irreducible sets Σ_J , $J \subset \{0, \dots, l\}$. Thus, we can reformulate Proposition 2.29 as follows: every set of the form Σ_J is contained in the closure of a set of the form Σ_J or $\tilde{S}_{A'_0, \dots, A'_l}$, whose codimension is 1.

For a generic point $(\varphi_0, \dots, \varphi_l) \in \Sigma_J$, we have the following three cases:

- 1) The set of all $y \in (\mathbb{C} \setminus 0)^k$, such that $d\varphi_0(y), \dots, d\varphi_l(y)$ are linearly dependent, has positive dimension. Then it contains a germ of a curve $(\mathbb{C} \setminus 0) \rightarrow (\mathbb{C} \setminus 0)^k$, whose leading term is $(c_1 t^{\gamma_1}, \dots, c_k t^{\gamma_k}) \neq \text{const}$. The covector $(\gamma_1, \dots, \gamma_k) \neq 0$, as a function on A_i , attains its maximum at some proper face $A'_i \subsetneq A_i$. Thus, $(\varphi_0, \dots, \varphi_l) \in \tilde{S}_{A'_0, \dots, A'_l}$, which is a hypersurface by induction.
- 2) $J = \{0, \dots, l\}$, and there exists an isolated point $y \in (\mathbb{C} \setminus 0)^k$, such that $d\varphi_0(y), \dots, d\varphi_l(y)$ are linearly dependent. Then we can choose vectors v_i near $d\varphi_i(y)$, such that the non-trivial linear combination of v_0, \dots, v_l is unique and has all non-zero coefficients. By Lemma 2.38, we can perturb the collection $(\varphi_0, \dots, \varphi_l)$ into $(\tilde{\varphi}_0, \dots, \tilde{\varphi}_l) \in \Sigma_J$, with $\tilde{\varphi}_i(y) = 0$ and $d\tilde{\varphi}_i(y) = v_i$. Thus, the polynomial $\sum_{i=0}^l \lambda_i \tilde{\varphi}_i \in \Sigma_{A_J}$ has an isolated line of singular points in its zero set, and $\Sigma_{A_J} = \Sigma_J$ is a hypersurface by Lemma 2.9.
- 3) In the general case, in the same way as above, we can perturb the collection $(\varphi_0, \dots, \varphi_l)$ into $(\tilde{\varphi}_0, \dots, \tilde{\varphi}_l) \in \Sigma_{\{0, \dots, l\}}$. But $\Sigma_{\{0, \dots, l\}}$ is either a hypersurface itself, or is contained in a hypersurface $\tilde{S}_{A'_0, \dots, A'_l}$ (see Cases 1 and 2). \square

2.8 Proof of GKZ decomposition formula.

To deduce Gelfand-Kapranov-Zelevinsky's decomposition formula (Proposition 2.10) from the relative Kouchnirenko-Bernstein formula, we reformulate the definition of the A -determinant as follows (we use notation and facts from Subsection 1.6).

GEOMETRIC CHARACTERIZATION OF A -RESULTANT AND A -DETERMINANT. Denote the projection $(\mathbb{C} \setminus 0)^k \times \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k] \rightarrow \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ by p . Let R_i be the tautological polynomial on $(\mathbb{C} \setminus 0)^k \times \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ that assigns the number $\varphi_i(y)$ to a point $(y, \varphi_0, \dots, \varphi_k) \in (\mathbb{C} \setminus 0)^k \times \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$. Denote the convex hull of A_i by B_i , and the dual fan of $B_0 + \dots + B_k$ by Σ . Then $s_{B_i} \cdot R_i$ extends to a section \tilde{R}_i of the line bundle I_{B_i} on the product $\mathbb{T}^\Sigma \times \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ (see Subsection 1.3 for the notation I_B and s_B), and we can reformulate the definition of the resultant R_{A_0, \dots, A_k} as follows.

LEMMA 2.43. $[R_{A_0, \dots, A_k} = 0] = p_*[\tilde{R}_0 = \dots = \tilde{R}_k = 0]$.

We also need a similar description for the A -determinant. Denote the projection $(\mathbb{C} \setminus 0)^k \times \mathbb{C}[A] \rightarrow \mathbb{C}[A]$ by p . Let S_0 be the tautological polynomial on $(\mathbb{C} \setminus 0)^k \times \mathbb{C}[A]$ that assigns the number $\varphi(y)$ to a point $(y, \varphi) \in (\mathbb{C} \setminus 0)^k \times \mathbb{C}[A]$, and let S_i be $y_i \frac{\partial S_0}{\partial y_i}$, where y_1, \dots, y_k are the standard coordinates on $(\mathbb{C} \setminus 0)^k$. Denote the convex hull of A by B and its dual fan by Σ . Then $s_B \cdot S_i$ extends to a section \tilde{S}_i of the line bundle I_B on the product $\mathbb{T}^\Sigma \times \mathbb{C}[A]$, and we can reformulate the definition of the A -determinant as follows.

LEMMA 2.44. $[E_A = 0] = p_*[\tilde{S}_0 = \dots = \tilde{S}_k = 0]$.

PROOF. Let j be the inclusion $\mathbb{C}[A] \hookrightarrow \underbrace{C[A] \oplus \dots \oplus C[A]}_{k+1}$ that assigns the collection $(\varphi, y_1 \frac{\partial \varphi}{\partial y_1}, \dots, y_k \frac{\partial \varphi}{\partial y_k})$ to every $\varphi \in \mathbb{C}[A]$, and denote the induced inclusion $(\mathbb{C} \setminus 0)^k \times \mathbb{C}[A] \hookrightarrow (\mathbb{C} \setminus 0)^k \times \underbrace{C[A] \oplus \dots \oplus C[A]}_{k+1}$ by the same letter j .

Then

$$\begin{aligned} p_*[\tilde{S}_0 = \dots = \tilde{S}_k = 0] &= p_* j^*[\tilde{R}_0 = \dots = \tilde{R}_k = 0] = \\ &= j^* p_*[\tilde{R}_0 = \dots = \tilde{R}_k = 0] = j^*[R_{A_0, \dots, A_k} = 0] = [E_A = 0], \end{aligned}$$

where the last two equalities are by Lemma 2.43 and by definition of the A -determinant respectively. \square

PROOF OF PROPOSITION 2.10. In the notation of Lemma 2.44, represent the complete intersection $[\tilde{S}_0 = \dots = \tilde{S}_k = 0]$ as a linear combination of irreducible varieties $\sum a_i V_i$. For every V_i , let \mathbb{T}_i be the minimal orbit of the toric variety \mathbb{T}^Σ , such that V_i is contained in the closure of $\mathbb{T}_i \times \mathbb{C}[A]$. For an arbitrary face A' of the set A , we denote the corresponding orbit of \mathbb{T}^Σ by $\mathbb{T}_{A'}$, and denote the sum $\sum_{i \mid \mathbb{T}_i = \mathbb{T}_{A'}} a_i V_i$ by $V_{A'}$. To prove the equality $[E_A = 0] = \sum_{A' \subset A} c^{A', A} [D_{A'} = 0]$ (which is exactly the statement of Proposition 2.10), it is enough to prove the following lemma.

LEMMA 2.45. $p_*(V_{A'}) = c^{A', A} [D_{A'} = 0]$.

PROOF. Denote $\dim A'$ by l , and choose a generic real $(k+1) \times (k+1)$ -matrix M , whose first $l+1$ rows generate the vector span of the set $\{1\} \times A' \subset \mathbb{Z} \oplus \mathbb{Z}^k$. Denote the entries of the product $M \cdot (\tilde{S}_0, \dots, \tilde{S}_k)^T$ by Z_0, \dots, Z_k (recall that they are sections of the line bundle I_B). The desired equality is a corollary of the following facts:

- 1) $[E_A = 0] = p_*([Z_0 = \dots = Z_{l-1} = 0] \cap [Z_l = \dots = Z_k = 0])$ by Lemma 2.44.
- 2) The orbit $\mathbb{T}_{A'}$ is a component of the complete intersection $[Z_0 = \dots = Z_{l-1} = 0]$ of multiplicity $c^{A', A}$ by Lemma 1.28.
- 3) The complete intersection $[Z_l = \dots = Z_k = 0]$ intersects the orbit $\mathbb{T}_{A'}$ transversally, and $p_*([Z_l = \dots = Z_k = 0] \cap \mathbb{T}_{A'}) = [D_{A'} = 0]$ by definition of the A -discriminant. \square

3 Eliminants.

In the first subsection, we formulate a local version of elimination theory in the context of Newton polyhedra (the global version is presented in [EKH]), i.e. we study the Newton polyhedron and leading coefficients of the equation of a projection of a complete intersection which is defined by equations with given Newton polyhedra and generic leading coefficients. The main result is Theorem 3.3, the proof is given in Subsection 3.2. In Subsection 3.3, we specialize this to A -determinants.

3.1 Elimination theory.

NOTATION. Let $\tau \subsetneq (\mathbb{R}^n)^*$ be a convex n -dimensional rational polyhedral strictly convex cone (i.e. a cone that does not contain a line), denote its dual cone $\{v \in \mathbb{R}^n \mid \gamma(v) \geq 0 \text{ for } \gamma \in \tau\}$ by τ^\vee , and consider the corresponding affine toric variety $\mathbb{T}^\tau = \text{spec } \mathbb{C}[\tau^\vee]$ with the maximal torus $(\mathbb{C} \setminus 0)^n$ and the vertex O . Note that τ^\vee is unbounded. A germ of a meromorphic function on \mathbb{T}^τ near O with no poles in the maximal torus can be represented as a power series $f(x) = \sum_{b \in \mathbb{Z}^n} c_b x^b$ for $x \in (\mathbb{C} \setminus 0)^n$, and the convex hull of the set $\{b \mid c_b \neq 0\} + \tau^\vee$ is called *the Newton polyhedron* Δ_f of f . The union of all bounded faces of Δ_f is called *the Newton diagram* $\partial\Delta_f$, and the coefficients c_b , $b \in \partial\Delta_f$, are called the *leading coefficients* of the germ f .

ELIMINANT. Let A_0, \dots, A_k be finite sets in \mathbb{Z}^k . For $i = 0, \dots, k$ and $a \in A_i$, let $f_{a,i}$ be a germ of a meromorphic function on the toric variety \mathbb{T}^τ with no poles in the maximal torus. We define the germs of functions F_0, \dots, F_k on $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k$ by the formula

$$F_i(x, y) = \sum_{a \in A_i} f_{a,i}(x) y^a \text{ for } x \in \mathbb{T}^\tau, y \in (\mathbb{C} \setminus 0)^k,$$

note that $F_i(x, \cdot) \in \mathbb{C}[A_i]$, and denote the number $R_{A_0, \dots, A_k}(F_0(x, \cdot), \dots, F_k(x, \cdot))$ by $R_{F_0, \dots, F_k}(x)$ (see Subsection 2.1 for the definition of the resultant R_{A_0, \dots, A_k}).

DEFINITION 3.1. The function R_{F_0, \dots, F_k} on the toric variety \mathbb{T}^τ is called *the eliminant of the projection of the complete intersection $F_0 = \dots = F_k = 0$ to \mathbb{T}^τ* .

Geometric meaning of the function R_{F_0, \dots, F_k} is as follows: if leading coefficients of the functions $f_{a,i}$ are in general position, and the image of the complete intersection $F_0 = \dots = F_k = 0$ under the projection $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k \rightarrow \mathbb{T}^\tau$ has codimension 1, then the closure of this image is the zero locus of R_{F_0, \dots, F_k} (see the beginning of the next subsection for details). Under these assumptions, the Newton polyhedron of R_{F_0, \dots, F_k} does not depend on coefficients of the functions $f_{a,i}$, but only on their Newton polyhedra. Theorem 3.3 below solves the following problem:

Express the Newton polyhedron and leading coefficients of the eliminant R_{F_0, \dots, F_k} in terms of the Newton polyhedra and leading coefficients of the functions $f_{a,i}$, provided that leading coefficients are in

general position, and describe this condition of general position explicitly.

CONDITION OF GENERAL POSITION. We denote the Newton polyhedron of $f_{a,i}$ by $\Delta_{a,i}$, and define the Newton polyhedron Δ_i of the function F_i as the convex hull of the set $\bigcup_{a \in A_i} \Delta_{a,i} \times \{a\} \subset \mathbb{R}^n \oplus \mathbb{R}^k$; then $F_i(z)$ can be represented as a power series $\sum_{b \in \Delta_i} c_{b,i} z^b$ for $z \in (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k$. If Γ is a face of Δ_i , then we denote the function $\sum_{b \in \Gamma} c_{b,i} z^b$ by F_i^Γ .

DEFINITION 3.2. The leading coefficients of the functions F_0, \dots, F_k are said to be *in general position*, if, for every collection of compatible bounded faces $\Gamma_i \subset \Delta_i$, $i = 0, \dots, k$ (see Definition 1.13), such that the restriction of the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$ to $\Gamma_0 + \dots + \Gamma_k$ is injective, the system of polynomial equations $F_0^{\Gamma_0} = \dots = F_k^{\Gamma_k} = 0$ has no solutions in $(\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k$.

This condition is obviously satisfied for generic leading coefficients of the functions $f_{a,i}$. Note that this condition is slightly weaker than the one in [EKH]. For example, if a face $\Gamma_i \subset \Delta_i$ is contained in a fiber of the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$, then we do not impose any assumptions on the leading coefficients of F_i , corresponding to internal integer points of Γ_i . This slight difference is important for our purpose (cf. [G]), see the proof of Proposition 3.7 below.

ELIMINATION THEOREM. We recall that the Minkowski integral $\int \Delta \subset \mathbb{R}^n$ of a polyhedron $\Delta \subset \mathbb{R}^n \oplus \mathbb{R}^k$ is defined in Section 1. For bounded faces $\Gamma_i \subset \Delta_i$, $i = 0, \dots, k$, we denote the intersection of A_i with the image of Γ_i under the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$ by $\tilde{\Gamma}_i$, and the value $R_{\tilde{\Gamma}_0, \dots, \tilde{\Gamma}_k} \left(F_0^{\Gamma_0}(x, \cdot), \dots, F_k^{\Gamma_k}(x, \cdot) \right)$ by $R_{F_0^{\Gamma_0}, \dots, F_k^{\Gamma_k}}(x)$ for $x \in (\mathbb{C} \setminus 0)^n$, then the Laurent polynomial $R_{F_0^{\Gamma_0}, \dots, F_k^{\Gamma_k}}$ on $(\mathbb{C} \setminus 0)^n$ depends only on leading coefficients of the functions $f_{a,i}$.

THEOREM 3.3. 1) *The Newton polyhedron of the eliminant R_{F_0, \dots, F_k} is contained in the mixed fiber polyhedron*

$$\text{MP}(\Delta_0, \dots, \Delta_k).$$

These two polyhedra coincide if and only if the leading coefficients of F_0, \dots, F_k are in general position in the sense of Definition 3.2.

2) *For every face Γ of the polyhedron $\text{MP}(\Delta_0, \dots, \Delta_k) \subset \mathbb{R}^n \subset \mathbb{R}^n \oplus \mathbb{R}^k$,*

$$R_{F_0, \dots, F_k}^\Gamma = \prod_{\Gamma_0, \dots, \Gamma_k} R_{F_0^{\Gamma_0}, \dots, F_k^{\Gamma_k}},$$

where the collection $(\Gamma_0, \dots, \Gamma_k)$ runs over all collections of faces of the polyhedra $\Delta_0, \dots, \Delta_k$, such that $\Gamma, \Gamma_0, \dots, \Gamma_k$ are compatible.

The proof of a global version of this theorem is given in [EKH], but cannot be extended to the local case word by word.

3.2 Proof of Theorem 3.3.

GEOMETRIC CHARACTERIZATION OF ELIMINANT. We can describe the geometric meaning of the eliminant R_{F_0, \dots, F_k} as follows (we assume that all functions $f_{a,i}$, $a \in A_i$, are holomorphic for simplicity). Denote the convex hull of A_i by B_i , and the dual fan of $B_0 + \dots + B_k$ by Σ . Then the product $s_{B_i} \cdot F_i$ extends to a section \tilde{F}_i of the line bundle I_{B_i} on the product $\mathbb{T}^\Sigma \times \mathbb{T}^\tau$ (we use the notation I_B , s_B and $m(f_1 \cdot \dots \cdot f_k \cdot V)$ introduced in Subsection 1.3). Denote the projection $\mathbb{T}^\Sigma \times \mathbb{T}^\tau \rightarrow \mathbb{T}^\tau$ by p .

LEMMA 3.4. *For a germ of a curve $C \subset \mathbb{T}^\tau$ near the origin,*

$$m(R_{F_0, \dots, F_k} \cdot C) = m(\tilde{F}_0 \cdot \dots \cdot \tilde{F}_k \cdot p^{(-1)}(C)).$$

In particular, both parts of the equality make sense simultaneously.

PROOF. We first consider the special case $R_{F_0, \dots, F_k} = R_{A_0, \dots, A_k}$. In this case $\mathbb{T}^\tau = \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$, and the function F_i equals the *tautological* function R_i on $(\mathbb{C} \setminus 0)^k \times \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$, that maps a point $(x, \varphi_0, \dots, \varphi_k)$ to $\varphi_i(x)$. Accordingly, we denote the section \tilde{F}_i by \tilde{R}_i in this case.

1) If $R_{F_0, \dots, F_k} = R_{A_0, \dots, A_k}$, and C intersects the set $R_{A_0, \dots, A_k} = 0$ transversally, then the statement follows by definition of the A -resultant (Definition 2.1).

2) If $R_{F_0, \dots, F_k} = R_{A_0, \dots, A_k}$, and C intersects the set $R_{A_0, \dots, A_k} = 0$ properly, then we can perturb C so that it intersects the set $R_{A_0, \dots, A_k} = 0$ transversally at a finitely many points, which reduces the statement to the case (1).

3) In general, consider the map $\mathcal{F} : \mathbb{T}^\tau \rightarrow \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ that assigns the collection of polynomials $(F_0(x, \cdot), \dots, F_k(x, \cdot))$ to every point $x \in \mathbb{T}^\tau$. Accordingly, denote the induced map $\mathbb{T}^\Sigma \times \mathbb{T}^\tau \rightarrow \mathbb{T}^\Sigma \times \mathbb{C}[A_0] \oplus \dots \oplus \mathbb{C}[A_k]$ by (id, \mathcal{F}) . Then we have

$$R_{F_0, \dots, F_k} = R_{A_0, \dots, A_k} \circ \mathcal{F} \quad \text{and} \quad \tilde{F}_i = (\text{id}, \mathcal{F})^* \tilde{R}_i,$$

which reduces the general case to the case (2). \square

PROOF OF THEOREM 3.3. First, we can assume without loss of generality that all germs $f_{a,i}$, $a \in A_i$, are holomorphic. Indeed, if we multiply every $f_{a,i}$ by a monomial x^{c_i} , where $c_i \in \tau^\vee$ is far enough from the boundary of τ^\vee , then all functions $f_{a,i}$ become holomorphic, while the statement of Theorem 3.3 does not change because of the homogeneity of the (A_0, \dots, A_k) -resultant.

PROOF OF PART 1. For an arbitrary positive integer linear function l on the cone τ^\vee , let (l_1, \dots, l_n) be its differential, pick generic complex numbers c_1, \dots, c_n , and consider the corresponding germ of a monomial curve $C : \mathbb{C} \rightarrow \mathbb{T}^{\tau^\vee}$ defined by the formula $C(t) = (c_1 t^{l_1}, \dots, c_n t^{l_n}) \in (\mathbb{C} \setminus 0)^n$ for $t \neq 0$. Then the minimal value of l on the Newton polyhedron of the eliminant R_{F_0, \dots, F_k} equals the intersection number $m(R_{F_0, \dots, F_k} \cdot C)$, which, by Lemma 3.4, equals

$$m(\tilde{F}_0 \cdot \dots \cdot \tilde{F}_k \cdot p^{(-1)}(C)).$$

Denote the convex hull of A_i by B_i , the projection $\mathbb{R}^k \oplus \mathbb{R}^n \rightarrow \mathbb{R}^k \oplus \mathbb{R}$ along $\ker l \subset \mathbb{R}^n$ by π_l , and the restriction of the germ F_i to the toric variety $p^{(-1)}(C)$ by

G_i . If the leading coefficients of the functions F_0, \dots, F_k are in general position in the sense of Definition 3.2, and the exponents l_1, \dots, l_n are generic in the sense that the restriction of the projection π_l to $\Delta_0 + \dots + \Delta_k$ is one-to-one over bounded faces of its image, then the Newton polyhedron of G_i equals $\pi_l(\Delta_i)$, and the leading coefficients of G_i are in general position in the sense of Definition 1.14. Thus, by Theorem 1.15, we have

$$m\left(\tilde{F}_0 \cdot \dots \cdot \tilde{F}_k \cdot p^{(-1)}(C)\right) = (k+1)! \text{MV}\left((\pi_l \Delta_0, B_0 \times \mathbb{R}_+), \dots, (\pi_l \Delta_k, B_k \times \mathbb{R}_+)\right).$$

Thus, if \mathcal{N} is the desired Newton polyhedron of the eliminant R_{F_0, \dots, F_k} , then, for every positive integer linear function l on the cone τ^\vee , we have

$$\min l|_{\mathcal{N}} = (k+1)! \text{MV}\left((\pi_l \Delta_0, B_0 \times \mathbb{R}_+), \dots, (\pi_l \Delta_k, B_k \times \mathbb{R}_+)\right).$$

This is exactly the formula for the support function of the mixed fiber polyhedron $\text{MP}(\Delta_0, \dots, \Delta_k)$, see Proposition 1.10.

REMARK. Suppose that, on the contrary, the condition of general position of Definition 3.2 is not satisfied for some compatible faces $\Gamma_0, \dots, \Gamma_k$ of the polyhedra $\Delta_0, \dots, \Delta_k$. Then the Newton polyhedron of the eliminant R_{F_0, \dots, F_k} is strictly smaller than the mixed fiber polyhedron $\text{MP}(\Delta_0, \dots, \Delta_k)$.

Namely, pick the face Γ of the polyhedron $\text{MP}(\Delta_0, \dots, \Delta_k)$, compatible with $\Gamma_0, \dots, \Gamma_k$. Consider a linear function l that attains its minimum on Γ as a function on $\text{MP}(\Delta_0, \dots, \Delta_k)$. Then, in the notation of the proof of Part 1, the leading coefficients of the functions G_0, \dots, G_k are not in general position in the sense of Definition 1.14, thus

$$\min l|_{\mathcal{N}} > (k+1)! \text{MV}\left((\pi_l \Delta_0, B_0 \times \mathbb{R}_+), \dots, (\pi_l \Delta_k, B_k \times \mathbb{R}_+)\right),$$

thus, the face Γ is not contained in the Newton polyhedron of the eliminant R_{F_0, \dots, F_k} .

PROOF OF PART 2. First, suppose that the condition of general position of Definition 3.2 is not satisfied for some faces $\Gamma_0, \dots, \Gamma_k$ of the polyhedra $\Delta_0, \dots, \Delta_k$, compatible with the face Γ . Then the corresponding factor in the right hand side of the desired equality vanishes. On the other hand, by the remark above, the left hand side vanishes as well.

Suppose that, on the contrary, the condition of general position of Definition 3.2 is satisfied for all collections of faces $\Gamma_0, \dots, \Gamma_k$ of the polyhedra $\Delta_0, \dots, \Delta_k$, compatible with the face Γ . Then the desired equality is proved in [EKh]. Note that the proof in [EKh] is written in the global setting, with a complex torus instead of the toric variety \mathbb{T}^τ , and polynomials instead of analytic functions on it. However, one can readily verify that the same proof remains valid in the local setting as well. \square

3.3 Reach discriminants and their Newton polyhedra.

The following version of the discriminant of a projection is not what we promised to study in the introduction; nevertheless, it allows us to reduce the study of

discriminants of projections to elimination theory. Let A be a finite set in \mathbb{Z}^k , and let f_a be a germ of a meromorphic function on the toric variety \mathbb{T}^τ for every $a \in A$. Define the germ of a function F on $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k$ by the formula

$$F(x, y) = \sum_{a \in A} f_a(x) y^a \text{ for } x \in \mathbb{T}^\tau, y \in (\mathbb{C} \setminus 0)^k,$$

and denote the number $E_A(F(x, \cdot))$ by $E_F(x)$ for every $x \in \mathbb{T}^\tau$ near the origin (see Subsection 2.1 for the definition of the A -determinant E_A).

DEFINITION 3.5. The function E_F on the toric variety \mathbb{T}^τ is called the *reach discriminant of the projection of the hypersurface $F = 0$ to \mathbb{T}^τ* .

We denote the Newton polyhedron of f_a by Δ_a , and define the Newton polyhedron Δ of the function F as the convex hull of the set $\bigcup_{a \in A} \Delta_a \times \{a\} \subset \mathbb{R}^n \oplus \mathbb{R}^k$; then $F(z)$ can be represented as a power series $\sum_{b \in \Delta} c_b z^b$ for $z \in (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k$. For any $\Gamma \subset \mathbb{R}^n \oplus \mathbb{R}^k$, we denote the function $\sum_{b \in \Gamma} c_b z^b$ by F^Γ .

DEFINITION 3.6. The leading coefficients of the functions f_a , $a \in A$, are said to be *in general position*, if, for every bounded face $\Gamma \subset \Delta$, such that the restriction of the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$ to Γ is injective, 0 is a regular value of the Laurent polynomial F^Γ .

Obviously, this condition is satisfied for generic leading coefficients of the functions f_a .

PROPOSITION 3.7. 1) *If the leading coefficients of the functions f_a , $a \in A$ are in general position in the sense of Definition 3.6, then the Newton polyhedron of E_F equals $\int \Delta$.*

2) *For every bounded face Γ of the polyhedron $\int \Delta \subset \mathbb{R}^n \subset \mathbb{R}^n \oplus \mathbb{R}^k$,*

$$E_F^\Gamma = \prod E_{F^{\Gamma'}},$$

where Γ' runs over all compatible with Γ bounded faces Γ' of the polyhedron $\Delta \subset \mathbb{R}^n \oplus \mathbb{R}^k$, such that the image of Γ' under the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$ has the same dimension as A .

PROOF. We can assume that $\dim A = k$ without loss of generality (otherwise, we can dehomogenize the function F). Consider $k+1$ generic linear combinations of the functions $F, y_1 \frac{\partial F}{\partial y_1}, \dots, y_k \frac{\partial F}{\partial y_k}$, where y_1, \dots, y_k are the standard coordinates on the torus $(\mathbb{C} \setminus 0)^k$. We denote these linear combinations by F_0, \dots, F_k , and note that Δ is the Newton polyhedron of each of these functions (while it is not always the Newton polyhedron of the functions $y_i \frac{\partial F}{\partial y_i}$). General position for the leading coefficients of the functions f_a , $a \in A$, in the sense of Definition 3.6 implies general position for the leading coefficients of the functions F_0, \dots, F_k in the sense of Definition 3.2, thus the statement of Proposition 3.7 follows from Theorem 3.3 for the functions F_0, \dots, F_k . \square

4 Discriminants of hypersurfaces.

In this section, we study the Newton polyhedron and leading coefficients of the discriminant of a projection of an analytic hypersurface, whose Newton polyhedron is given and whose leading coefficients are in general position.

In the first subsection we give an “algebraic” definition of the discriminant, and clarify its geometric meaning in Propositions 4.2 and 4.3; the proof of these facts occupies Subsections 4.2 and 4.3. In Subsection 4.4, we study the Newton polyhedron (Theorem 4.10) and leading coefficients (Proposition 4.11) of the discriminant. These results are proved in the last subsection.

4.1 Discriminants of hypersurfaces.

Let A be a finite set in \mathbb{Z}^k , let $\tau \subset (\mathbb{R}^n)^*$ be a convex n -dimensional rational polyhedral cone that does not contain a line, and let f_a be a germ of a meromorphic function on the affine toric variety \mathbb{T}^τ for every $a \in A$. Define the germ of a function F on $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k$ by the formula

$$F(x, y) = \sum_{a \in A} f_a(x) y^a \text{ for } x \in \mathbb{T}^\tau, y \in (\mathbb{C} \setminus 0)^k,$$

note that $F(x, \cdot) \in \mathbb{C}[A]$, and denote the number $D_A(F(x, \cdot))$ by $D_F(x)$ (see Subsection 2.2 for the definition of the A -discriminant D_A).

DEFINITION 4.1. The function D_F on the toric variety \mathbb{T}^τ is called *the discriminant of the projection of the hypersurface $F = 0$ to \mathbb{T}^τ* .

The discriminant has the expected geometric meaning if the leading coefficients are in general position. Namely, denote the set $\{x \in (\mathbb{C} \setminus 0)^n \mid D_F(x) = 0\}$ by $Z(F)$, and consider the set $\Sigma(F)$ of all $x \in (\mathbb{C} \setminus 0)^n$ such that 0 is a singular value of the polynomial $F(x, \cdot)$ on $(\mathbb{C} \setminus 0)^k$.

PROPOSITION 4.2. *Suppose that the Newton polyhedra of the functions f_a , $a \in A$, are given, and the leading coefficients of these functions are in general position. If A is not dual defect (for example, if A satisfies assumptions of Proposition 2.14), then $\overline{\Sigma(F)} = \overline{Z(F)}$ in $(\mathbb{C} \setminus 0)^n$, otherwise $\text{codim } \Sigma(F) > 1$.*

This statement can be extended from the maximal torus $(\mathbb{C} \setminus 0)^n$ to the toric variety \mathbb{T}^τ as follows. For a face θ of the cone $\tau^\vee \subset \mathbb{R}^n$, define $A(\theta)$ as the set of all a such that the Newton polyhedron of f_a intersects θ . Consider the set $\Sigma_0(F)$ of all $x \in \mathbb{T}^\tau$ such that 0 is a singular value of the polynomial $F(x, \cdot)$ on $(\mathbb{C} \setminus 0)^k$.

PROPOSITION 4.3. *Suppose that the functions f_a , $a \in A$, are holomorphic, their Newton polyhedra are such that $A(\theta) \neq \emptyset$ for every codimension 1 face $\theta \subset \tau^\vee$, and their leading coefficients are in general position. Then*
1) the union of all codimension 1 components of $\overline{\Sigma_0(F)}$ equals $\overline{Z(F)}$ in \mathbb{T}^τ .
2) If, in addition, A is not dual defect and $\dim A(\theta) > \dim A + \dim \theta - n$ for every $\theta \neq \tau^\vee$, then $\overline{\Sigma_0(F)} = \overline{Z(F)}$ (in particular, $\overline{\Sigma_0(F)}$ is a hypersurface).

Note that $\overline{Z(F)}$ is contained in the zero set of the discriminant D_F on \mathbb{T}^τ , but is smaller in general (even under the assumptions of the proposition). The equality of Proposition 4.3(2) may turn into the strict inequality $\overline{\Sigma_0(F)} \subsetneq \overline{Z(F)}$ in the case of arbitrary leading coefficients of the functions f_a , and even this inequality may be not valid if $\dim A(\theta) < \dim A + \dim \theta - n$ for some θ . One can readily observe corresponding examples in the simplest non-trivial case $n = 1$, $A = \{0, 1, 2\} \subset \mathbb{Z}^1$; a more refined example with $\dim A(\theta) = \dim A + \dim \theta - n$ and $\Sigma(F)$ not of pure dimension is given at the end of Subsection 4.3.

4.2 Maps with generic leading coefficients.

To prove the above statements, we need the following

PROPOSITION 4.4. *Let h_1, \dots, h_p be either*

- 1) *Laurent polynomials on the complex torus $(\mathbb{C} \setminus 0)^n$, or*
- 2) *germs of meromorphic functions on an affine toric variety (\mathbb{T}^τ, O) with no poles in the maximal torus $(\mathbb{C} \setminus 0)^n$.*

In both cases, consider the map $h = (h_1, \dots, h_p) : (\mathbb{C} \setminus 0)^n \rightarrow \mathbb{C}^p$.

If $S \subset \mathbb{C}^p$ is an arbitrary algebraic set of codimension s , the Newton polyhedra of the functions h_i are given, and the leading coefficients of these functions are in general position, then the set $h^{(-1)}(S)$ has the same codimension s .

Note that, in both settings, h is defined as a map from the torus $(\mathbb{C} \setminus 0)^n$, rather than from the toric variety (\mathbb{T}^τ, O) , and, in particular, $h^{(-1)}(S) \subset (\mathbb{C} \setminus 0)^n$. If $S' \subset \mathbb{C}^p$ is a constructible set (i.e. if it is obtained by applying the operations of union, intersection and subtraction to algebraic sets), then, applying Proposition 4.4 to its closure $\overline{S'}$ and to the closure of the difference $\overline{S'} \setminus S'$, one gets

COROLLARY 4.5. *If $S' \subset \mathbb{C}^p$ is a constructible set, then, under the assumptions of Proposition 4.4, the closure of $h^{(-1)}(S')$ equals $h^{(-1)}(\overline{S'})$.*

To prove Proposition 4.4, we reduce it to the following lemma.

LEMMA 4.6. *Let g_1, \dots, g_p be Laurent polynomials on $(\mathbb{C} \setminus 0)^k$, whose coefficients are germs of meromorphic functions on an affine toric variety (\mathbb{T}^τ, O) with no poles in its maximal torus $(\mathbb{C} \setminus 0)^n$. If $Q \subset (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k$ is an arbitrary algebraic set, the Newton polyhedra of the functions g_i are given, and the leading coefficients of these functions are in general position, then the set $\{g_1 = \dots = g_p = 0\}$ intersects Q properly near $\{O\} \times (\mathbb{C} \setminus 0)^k \subset \mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k$.*

DEFINITION 4.7. For a covector $\gamma \in (\mathbb{Z}^n)^*$ and an analytic function $g(y) = \sum_{a \in \mathbb{Z}^n} c_a y^a$ on $(\mathbb{C} \setminus 0)^n$, the γ -truncation g^γ is defined to be the last non-zero sum in the sequence of sums $\sum_{a \mid \gamma(a)=k} c_a y^a$, $k \in \mathbb{Z}$, provided that these sums are equal to 0 for large k .

For a covector $\gamma \in (\mathbb{Z}^n)^*$ and an ideal I in $\mathbb{C}[\mathbb{Z}^n]$, the γ -truncation I^γ is the ideal, generated by γ -truncations of all elements of I . The γ -truncation Q^γ of an algebraic variety $Q \in (\mathbb{C} \setminus 0)^n$ is defined to be the zero locus of the γ -truncation of its ideal.

PROOF OF LEMMA 4.6. Choose any covector $\gamma \in (\mathbb{R}^n \times \mathbb{R}^k)^*$ that takes only negative values on the closed cone $\tau^\vee \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^k$. By the Bertini-Sard theorem, the set $\{g_1^\gamma = \dots = g_p^\gamma = 0\}$ intersects Q^γ properly under an appropriate assumption of general position for the leading coefficients of the functions g_1, \dots, g_p . Since the set of all possible varieties of the form Q^γ is finite (see e.g. [S96] or [Kaz]), one can choose the latter assumption of general position to be independent of γ . Under this assumption, the set $\{g_1^\gamma = \dots = g_p^\gamma = 0\}$ intersects Q^γ properly for every γ , thus the same holds for $\{g_1 = \dots = g_p = 0\}$ and Q near the set $\{O\} \times (\mathbb{C} \setminus 0)^k \subset \mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k$ (see e.g. [S96] or [Kaz]). \square

In the same way, if Q is smooth, one can prove that $\{g_1 = \dots = g_p = 0\}$ intersects Q transversally, if leading coefficients are in general position.

PROOF OF PROPOSITION 4.4. Denote the standard coordinates on \mathbb{C}^p by y_1, \dots, y_p , consider an arbitrary subdivision $\{1, \dots, p\} = I \sqcup J$, and denote the torus $\{(y_1, \dots, y_p) \mid y_i \neq 0 \text{ for } i \in I \text{ and } y_j = 0 \text{ for } j \in J\}$ by $(\mathbb{C} \setminus 0)^I$, and apply the following Lemma 4.6 to the set $Q = (S \cap (\mathbb{C} \setminus 0)^I) \times \mathbb{T}^\tau$ and functions $g_i = \begin{cases} h_i - y_i & \text{for } i \in I \\ h_i & \text{for } i \in J \end{cases}$ on the toric variety $(\mathbb{C} \setminus 0)^I \times \mathbb{T}^\tau$. \square

4.3 Proof of Propositions 4.2 and 4.3.

PROOF OF PROPOSITION 4.2. Recall that $\Sigma \subset \mathbb{C}[A]$ is the set of all φ such that $\varphi(y) = d\varphi(y) = 0$ for some $y \in (\mathbb{C} \setminus 0)^k$. Its closure $\overline{\Sigma}$ is a hypersurface and is defined by the equation $D_A = 0$ (otherwise identically $D_A = 1$ and $Z(F) = \emptyset$ by definition).

Let $\mathcal{F} : \mathbb{T}^\tau \rightarrow \mathbb{C}[A]$ be the map that assigns the polynomial $F(x, \cdot)$ to a point $x \in \mathbb{T}^\tau$. We can express the desired sets $\Sigma(F)$ and $Z(F) = \{D_F = 0\}$ in terms of this map:

$$\Sigma(F) = \mathcal{F}^{(-1)}(\Sigma) \cap (\mathbb{C} \setminus 0)^n,$$

$$D_F = D_A \circ \mathcal{F}.$$

Proposition 4.2 now follows by Corollary 4.4 with $(h_1, \dots, h_p) = \mathcal{F}$ and $S' = \Sigma$. \square

We can also formulate a refinement of Proposition 4.2 with multiplicities taken into account (the proof follows the same lines but requires more technical details; we omit it, since we do not need this refinement in what follows). Restrict the projection $p : \mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k \rightarrow \mathbb{T}^\tau$ to the regular locus of the set $\{F = 0\}$ and denote the singular locus of this restriction by S .

PROPOSITION 4.8. *(See Subsection 1.6 for the notation.)*
 $p_*(S)$ equals $[D_{\overline{F}} = 0]$ on the complex torus $(\mathbb{C} \setminus 0)^n$, and does not contain codimension 1 orbits of the toric variety \mathbb{T}^τ , if the leading coefficients of the functions f_a , $a \in A$, are in general position.

PROOF OF PROPOSITION 4.3. For a closed regular subvariety $M \subset (\mathbb{C} \setminus 0)^k$, define $\Sigma(M) \subset \mathbb{C}[A]$ as the set of all Laurent polynomials φ in $\mathbb{C}[A]$, such that the set $\{y \in M \mid \varphi(y) = d\varphi(y) = 0\}$ has at least one isolated point. Such definition implies the following properties of $\Sigma(M)$ (in contrast to Σ):

1) for every $\varphi \in \Sigma(M)$, at least one of the local components of $\Sigma(M)$ near φ is

closed.

2) The set $\Sigma(M)$ has codimension $1 + \text{codim } M + \dim A - k$ at all of its points. We need the following corollary of (1) and (2):

3) The set $\mathcal{F}^{(-1)}(\Sigma(M))$ has codimension at most $1 + \text{codim } M + \dim A - k$ at all of its points. We denote the latter set by $\Sigma(M, F)$.

For every face $\theta \subset \tau^\vee$, we can choose a closed regular subvariety $M_\theta \subset (\mathbb{C} \setminus 0)^k$ such that $\Sigma(M_\theta) \cap \mathbb{C}[A(\theta)]$ is dense in $\Sigma \cap \mathbb{C}[A(\theta)]$. Let T_θ be the orbit of the variety \mathbb{T}^τ , corresponding to the face θ , then the restriction of the map \mathcal{F} to this orbit is a map $\mathcal{F}_\theta : T_\theta \rightarrow \mathbb{C}[A(\theta)]$. Proposition 4.4 for $(h_1, \dots, h_p) = \mathcal{F}_\theta$, $S = \overline{\Sigma(M)}_\theta \cap \mathbb{C}[A(\theta)]$ and Corollary 4.5 for $(h_1, \dots, h_p) = \mathcal{F}_\theta$, $S = \Sigma(M)_\theta \cap \mathbb{C}[A(\theta)]$ imply the following:

4) If the leading coefficients of the functions f_a , $a \in A$, are in general position, then the set $\Sigma(M, F) \cap T_\theta$ is dense in $\Sigma_0(F) \cap T_\theta$, and its codimension in T_θ is equal to $1 + \text{codim } M + \dim A(\theta) - k$, which is greater than $1 + \text{codim } M + \dim A - k + \dim \theta - n$ under the assumption of Proposition 4.3.

Since, by (3) and (4), the codimension of $\Sigma(M, F) \cap T_\theta$ in the toric variety \mathbb{T}^τ is greater than the codimension $\Sigma(M, F)$ in \mathbb{T}^τ at every point of $\Sigma(M, F) \cap T_\theta$, then $\Sigma(M, F) \cap T_\theta$ is contained in the closure of $\Sigma(M, F) \setminus T_\theta$. The inclusions

$$\overline{\Sigma_0(F) \cap T_\theta} = \overline{\Sigma(M, F) \cap T_\theta} \subset \overline{\Sigma(M, F) \setminus T_\theta} \subset \overline{\Sigma(F)} \subset \overline{Z(F)}$$

prove Proposition 4.3. \square

EXAMPLE 4.9. Note that the inclusion $\overline{\Sigma_0(F) \cap T_\theta} \subset \overline{\Sigma(F)}$ and the statement of Proposition 4.3 may fail, if $\dim A(\theta) = \dim A + \dim \theta - n$. For example, let \mathbb{T}^τ be the space \mathbb{C}^3 with coordinates x, y, z , and let T_θ be the torus $\{x \neq 0, y \neq 0, z = 0\}$. Pick generic linear functions l_1, l_2, l_3, m_1, m_2 of the variables x and y , and choose the face A and the functions f_a , $a \in A$ as follows (each function $f_a(x, y, z)$ is written near the corresponding point $a \in A$):

$$A \left\{ \begin{array}{ccc} \bullet & & \bullet \\ z m_1(x, y) & z m_2(x, y) & \\ l_1(x, y) & l_2(x, y) & l_3(x, y) \\ \bullet & \bullet & \bullet \end{array} \right\} A(\theta)$$

Then the sets $\overline{\Sigma_0(F) \cap T_\theta}$ and $\overline{\Sigma(F)} \cap T_\theta$ are given by the equations

$$l_1^2 - l_1 l_3 = 0 \quad \text{and} \quad l_1 m_1^2 - l_2 m_1 m_2 + l_3 m_2^2 = 0$$

in x and y , and hence do not intersect.

4.4 Newton polyhedra of discriminants of hypersurfaces.

If the Newton polyhedra Δ_a of functions f_a , $a \in A \subset \mathbb{Z}^k$, on the affine toric variety \mathbb{T}^τ are given, and the leading coefficients of these functions satisfy a certain condition of general position, then the Newton polyhedron of the discriminant D_F , where $F(x, y) = \sum_{a \in A} f_a(x) y^a$, depends only on the Newton polyhedra of these functions, not on the coefficients. Theorem 4.10 and Proposition 4.11 below solve the following problem:

Express the Newton polyhedron and leading coefficients of the discriminant D_F in terms of the Newton polyhedra and leading coefficients of the functions f_a , provided that leading coefficients are in general position.

NEWTON POLYHEDRON OF THE DISCRIMINANT. The Minkowski integral $\int \Delta \subset \mathbb{R}^n$ of a polyhedron $\Delta \subset \mathbb{R}^n \oplus \mathbb{R}^k$ and combinatorial Euler obstructions $e^{A',A}$ and Milnor numbers $c^{A',A}$ are introduced in Section 1. Recall that a face of a set $A \subset \mathbb{Z}^k$ is the intersection of A with a face of its convex hull, and $\dim A$ is the dimension of its convex hull. For a face A' of the set $A \subset \mathbb{Z}^k$, we denote the convex hull of the union $\bigcup_{a \in A'} \Delta_a \times \{a\}$ by $\Delta(A')$; this is an unbounded face of $\Delta = \Delta(A)$.

THEOREM 4.10. *1) If the Newton polyhedra of the functions f_a are given, and the leading coefficients of these functions are in general position in the sense of Definition 3.6, then the Newton polyhedron of the discriminant D_F equals*

$$\mathcal{N}_\Delta^A = \sum_{A' \subset A} e^{A',A} \cdot \int \Delta(A'),$$

where A' runs over all faces of A , including $A' = A$.

2) If the leading coefficients of the functions f_a are arbitrary, then the Newton polyhedron of the discriminant D_F is contained in \mathcal{N}_Δ^A .

The proof is given in Subsection 4.5, an example of application is given in Subsection 5.6. Note that Definition 3.6 is not the weakest possible condition of general position for this theorem. The weakest one (very complicated to verify though) can be extracted from Proposition 4.11 below, see the subsequent discussion.

LEADING COEFFICIENTS OF THE DISCRIMINANT. Let Γ be a face of the polyhedron Δ , denote the set of all $a \in A$, such that Γ intersects $\Delta_a \times a$, by A_Γ , and denote the minimal face of A , containing A_Γ , by \bar{A}_Γ . For a bounded face $\tilde{\Gamma}$ of another polyhedron, define the number $d^{\tilde{\Gamma},\Gamma}$ as $\sum_{\Gamma'} c^{A_\Gamma, A_{\Gamma'}} \cdot e^{\bar{A}_{\Gamma'}, A}$, where Γ' runs over all bounded compatible with $\tilde{\Gamma}$ faces of Δ such that $\Gamma' \supset \Gamma$ and $\dim A_{\Gamma'} = \dim \bar{A}_{\Gamma'}$. Defining the value $D_{F\Gamma}(x)$ as $R_{A_\Gamma}(F^\Gamma(x, \cdot))$ for $x \in (\mathbb{C} \setminus 0)^n$, the Laurent polynomial $D_{F\Gamma}$ on $(\mathbb{C} \setminus 0)^n$ depends only on leading coefficients of the functions f_a , $a \in A_\Gamma$.

PROPOSITION 4.11. *1) For every bounded face $\tilde{\Gamma}$ of the expected Newton polyhedron \mathcal{N}_Δ^A of the discriminant D_F ,*

$$D_F^{\tilde{\Gamma}} = \prod_{\Gamma} (D_{F\Gamma})^{d^{\tilde{\Gamma},\Gamma}},$$

where Γ runs over all bounded faces of the polyhedron Δ .

2) Every factor $(D_{F\Gamma})^{d^{\tilde{\Gamma},\Gamma}}$ in the right hand side of this equality is a polynomial (i.e. $d^{\tilde{\Gamma},\Gamma} \geq 0$ whenever the polynomial $D_{F\Gamma}$ is of positive degree).

In particular, the leading coefficients of the discriminant D_F only depend on those of the functions f_a , $a \in A$. The proof is given in Subsection 4.5.

An assumption of general position for leading coefficients would be redundant in this statement: if the leading coefficients of the functions f_a , $a \in A$, are degenerate enough, then both parts of the equality become identically zero simultaneously. In particular, the Newton polyhedron of D_F equals \mathcal{N}_Δ^A (i.e. D_F has non-zero coefficients of the monomials, corresponding to the vertices of \mathcal{N}_Δ^A) if and only if the following condition is satisfied for all $\tilde{\Gamma}$ and Γ : if $\tilde{\Gamma}$ is a vertex, and $d^{\tilde{\Gamma}, \Gamma} > 0$, then $D_{F\Gamma}$ is not identically zero. Note that the coefficient $d^{\tilde{\Gamma}, \Gamma}$ is complicated to compute, and no simple combinatorial criterion for its positivity (i.e. for divisibility $D_F^{\tilde{\Gamma}}$ by $D_{F\Gamma}$) is known. See, for example, Theorem 15 in [CC] for one important special case.

DEGREE OF A -DISCRIMINANTS. The Gelfand-Kapranov-Zelevinsky discriminant D_A is a special case of the discriminant D_F with $\mathbb{T}^\tau = \mathbb{C}[A]$. In this case, the discriminant is homogeneous, and its Newton polyhedron \mathcal{N} is contained in the space \mathbb{R}^A , whose integer lattice consists of monomials of the form $\prod_{a \in A} c_a^{\lambda_a}$ in coefficients of the indeterminate polynomial $\sum_{a \in A} c_a x^a \in \mathbb{C}[A]$. We consider λ_a , $a \in A$, as a system of coordinates on \mathbb{R}^A , denote $\sum_a \lambda_a$ by l , and note that the degree of the discriminant D_A is the minimal value of l on \mathcal{N} .

Computing \mathcal{N} by Theorem 4.10 and then $\min l|_{\mathcal{N}}$ by Proposition 1.10 in this case, we get the following formula for the degree of D_A . Let A' be a face of A , and let $M \subset \mathbb{R}^k$ be the vector space, parallel to the affine span of A' . We choose the volume form μ on M such that the volume of $M/(M \cap \mathbb{Z}^k)$ equals $(\dim M)!$, and denote the μ -volume of the convex hull of A' by $\text{Vol } A'$.

DEFINITION 4.12. Define the number $\deg A$ as the sum

$$\sum_{A' \subset A} e^{A', A} \cdot (\dim A' + 1) \text{Vol}(A')$$

over all faces A' of the set A , including $A' = A$.

COROLLARY 4.13 ([MT]). 1) $\deg D_A = \deg A$.
2) D_A is a constant if and only if $\deg A = 0$.

REMARK. A useful generalization of this fact for discriminants of higher codimension is proved in [MT], based on a totally different technique (the Ernström formula [E]). Amazingly, that technique also ends up with Euler obstructions of toric varieties, which suggests that the two techniques could be unified. In particular, it would be interesting to find a common generalization of Theorem 4.10 and the Ernström formula, which would, for instance, compute the tropicalization of the dual of an arbitrary projective variety V in terms of Euler obstructions of truncations of V .

4.5 Proof of Theorem 4.10 and Proposition 4.11.

Applying the Gelfand-Kapranov-Zelevinsky decomposition (Proposition 2.10) to the definition of reach discriminant (Definition 3.5), we have the following relation

between discriminants and reach discriminants (see Definition 3.5) for every face $A' \subset A$:

$$E_{F\Delta(A')} = \prod_{A''} (D_{F\Delta(A'')})^{c^{A'', A'}}, \quad (*_{A'})$$

where A'' runs over all faces of A' , including $A'' = A'$. Inverting the formulas $(*_{A'})$ by induction on the dimension of A' , we have the following relations:

$$D_{F\Delta(A')} = \prod_{A''} (E_{F\Delta(A'')})^{e^{A'', A'}}, \quad (**_{A'})$$

where A'' runs over all faces of A' , including $A'' = A'$.

In more detail, assume that we have already obtained the formulas $(**_B)$ for all faces B of dimension less than p . Then, for every A' of dimension p , rewriting the formula $(*_{A'})$ as $D_{F\Delta(A')} = E_{F\Delta(A')} \cdot \prod_{A'' \neq A'} (D_{F\Delta(A'')})^{-c^{A'', A'}}$, expressing $D_{F\Delta(A'')}$ in terms of $E_{F\Delta(A'')}$ by the formulas $(**_B)$ in the right hand side, and collecting similar multipliers, we obtain the formula $(**_{A'})$.

Informally speaking, if we consider the formal logarithm of the formulas $(*_{A'})$ to pass to the additive notation instead of the multiplicative one, then the vector of logarithms $\ln E_{F\Delta(A')}$, $A' \subset A$, equals the matrix with entries $c^{A'', A'}$ times the vector of logarithms $\ln D_{F\Delta(A')}$, $A' \subset A$. Inverting the matrix, we obtain the logarithm of the formulas $(**_{A'})$.

$$\text{In particular, } D_A(\varphi) = \prod_{A'} (E_{A'}(\varphi^{A'}))^{e^{A', A}}.$$

PROOF OF THEOREM 4.10. To prove Part 1, apply Proposition 3.7(1) and the identity $(**_A)$ above.

To prove Part 2, consider a monomial curve $C : \mathbb{C} \rightarrow \mathbb{T}^\tau$, corresponding to an arbitrary positive integer linear function $\gamma : \tau^\vee \rightarrow \mathbb{R}_{>0}$; by definition, $C(t) = (h_1 t^{\gamma_1}, \dots, h_n t^{\gamma_n}) \in (\mathbb{C} \setminus 0)^n$, where $\gamma_1, \dots, \gamma_n$ are the coefficients of the linear function γ , and the coefficients h_1, \dots, h_n are generic. Then $D_F(C(t))$ is a germ of a meromorphic function of one variable t , and the order of its zero or pole is equal to the minimal value of the function γ on the Newton polyhedron of D_F . Since this order of zero depends upper-semicontinuously on F , so does the Newton polyhedron of D_F . \square

PROOF OF PROPOSITION 4.11, PART 1. For every face A' of the set A , choose a face $\tilde{\Gamma}(A')$ of the polyhedron $\int \Delta(A')$, such that these faces $\tilde{\Gamma}(A')$ together with $\tilde{\Gamma}$ form a compatible collection. Represent the discriminant D_F as a product of reach discriminants by formula $(**_A)$ above, then

$$D_F^{\tilde{\Gamma}} = \prod_{A'} (E_{F\Delta(A')}^{\tilde{\Gamma}(A')})^{e^{A', A}}.$$

By Proposition 3.7(2), represent every truncated reach discriminant $E_{F\Delta(A')}^{\tilde{\Gamma}(A')}$ in the right hand side as a product of reach discriminants of the form $E_{F\Gamma'}$, $\tilde{\Gamma}' \subset \Delta(A')$. Finally, represent each of these reach discriminants as a product of discriminants by formula $(*_{A_{\Gamma'}})$ above. \square

PROOF OF PROPOSITION 4.11, PART 2. If the set A_Γ is dual defect, then identically $D_{F\Gamma} = 1$, and the sign of the exponent $d^{\tilde{\Gamma}, \Gamma}$ in the right hand side of

the identity in the statement of Proposition 4.11(1) is not important. Otherwise, we have

LEMMA 4.14. *If the set A_Γ is not dual defect, then $d^{\tilde{\Gamma}, \Gamma} \geq 0$ for every bounded face $\tilde{\Gamma} \subset \mathcal{N}_\Delta^A$.*

PROOF. Let A_Γ consist of points a_1, \dots, a_N , and let $D_{A_\Gamma}(u_1, \dots, u_N)$ be the value of the discriminant D_{A_Γ} at the polynomial $\sum_i u_i t^{a_i} \in \mathbb{C}[A_\Gamma]$. Consider both sides of the identity in the statement of Proposition 4.11(1) as polynomials of the leading coefficients of the functions f_a , $a \in A$, with a fixed value of the variable $x \in (\mathbb{C} \setminus 0)^n$. Then the discriminant D_{F^Γ} equals $D_{A_\Gamma}(l_1, \dots, l_N)$, where l_i is a non-zero linear combination of leading coefficients of the function f_{a_i} . Since D_{A_Γ} is a power of a homogeneous irreducible polynomial that non-trivially depends on all the variables u_1, \dots, u_N (by Lemmas 2.8 and 2.21), so does $D_{F^\Gamma} = D_{A_\Gamma}(l_1, \dots, l_N)$: it is a power of a homogeneous irreducible polynomial that non-trivially depends on all the coefficients of F^Γ and does not depend on other leading coefficients of the functions f_a , $a \in A$.

Thus, if $d^{\tilde{\Gamma}, \Gamma}$ were negative for some Γ , then the factor $(D_{F^\Gamma})^{d^{\tilde{\Gamma}, \Gamma}}$ could not be cancelled by other multipliers in the right hand side of the equality in the statement of Proposition 4.11(1), and the right hand side were a rational function with a non-trivial denominator. But this is impossible because the left hand side is a polynomial. \square

5 Discriminants of complete intersections.

In this section, we generalize the results of the previous two sections to discriminants of projections of analytic complete intersections.

The discriminant is defined in the first subsection. In Subsection 5.2, we reduce the study of its Newton polyhedron and leading coefficients to the case of projections of hypersurfaces, which is studied in the previous section. In some important special cases (Subsections 5.3 and 5.4), this leads to an explicit answer. An example of computation of such answer is given in the last subsection. We also consider an alternative definition of the discriminant in Subsection 5.5.

5.1 Discriminants of complete intersections.

Let $\tau \subset (\mathbb{R}^n)^*$ be a convex n -dimensional rational polyhedral cone that does not contain a line, and let A_0, \dots, A_l , $l \leq k$, be finite sets in \mathbb{Z}^k . For every $i = 0, \dots, l$, $a \in A_i$, let $f_{a,i}$ be a germ of a meromorphic function on the toric variety \mathbb{T}^τ with no poles in the maximal torus. Define the germ of a function F_i on $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k$ by the formula

$$F_i(x, y) = \sum_{a \in A_i} f_{a,i}(x) y^a \text{ for } x \in \mathbb{T}^\tau, y \in (\mathbb{C} \setminus 0)^k,$$

and denote the number $D_{A_0, \dots, A_l}^{red}(F_0(x, \cdot), \dots, F_l(x, \cdot))$ by $D_{F_0, \dots, F_l}^{red}(x)$ (see Subsection 2.4 for the definition of the discriminant $D_{A_0, \dots, A_l}^{red}$).

DEFINITION 5.1. The germ of the function $D_{F_0, \dots, F_l}^{\text{red}}$ on the affine toric variety (\mathbb{T}^τ, O) is called *the discriminant of the projection of the complete intersection $F_0 = \dots = F_l = 0$ to \mathbb{T}^τ* .

The discriminant has the expected geometric meaning if the leading coefficients are in general position. Namely, denote the set $\{x \in (\mathbb{C} \setminus 0)^n \mid D_{F_0, \dots, F_l}^{\text{red}}(x) = 0\}$ by $Z(F_0, \dots, F_l)$, and consider the set $\Sigma(F_0, \dots, F_l)$ of all $x \in (\mathbb{C} \setminus 0)^n$, such that $(0, \dots, 0)$ is a singular value of the map $(F_0(x, \cdot), \dots, F_l(x, \cdot)) : (\mathbb{C} \setminus 0)^k \rightarrow \mathbb{C}^{l+1}$.

PROPOSITION 5.2. *Suppose that the Newton polyhedra of the functions $f_{a,i}$, $a \in A_i$, are given, and the leading coefficients of these functions are in general position. Then*

1) the union of codimension 1 components of the closure $\overline{\Sigma(F_0, \dots, F_l)}$ equals $\overline{Z(F_0, \dots, F_l)}$.

2) If, in addition, the collection A_0, \dots, A_k is not dual defect (for example, if it satisfies assumptions of Proposition 2.24), then

$$\overline{\Sigma(F_0, \dots, F_l)} = \overline{Z(F_0, \dots, F_l)},$$

and, in particular, $\Sigma(F_0, \dots, F_l)$ is a hypersurface.

This can be extended from the maximal torus $(\mathbb{C} \setminus 0)^n$ to the toric variety \mathbb{T}^τ in the same way as Proposition 4.2 (see Proposition 4.3 for the notation). Consider the set $\Sigma_0(F_0, \dots, F_l)$ of all $x \in \mathbb{T}^\tau$, such that $(0, \dots, 0)$ is a singular value of the map $(F_0(x, \cdot), \dots, F_l(x, \cdot)) : (\mathbb{C} \setminus 0)^k \rightarrow \mathbb{C}^{l+1}$.

PROPOSITION 5.3. *Suppose that the functions $f_{a,i}$, $a \in A_i$, are holomorphic, their Newton polyhedra are such that $A_j(\theta) \neq \emptyset$ for every codimension 1 face $\theta \subset \tau^\vee$ and $j = 0, \dots, l$, and their leading coefficients are in general position. Then*

1) the union of all codimension 1 components of $\overline{\Sigma_0(F_0, \dots, F_l)}$ equals $\overline{Z(F_0, \dots, F_l)}$.

2) If, in addition, the collection A_0, \dots, A_k is not dual defect, and $\dim \sum_j A_j(\theta) > \dim L + \dim \theta - n$ for every $\theta \neq \tau^\vee$, then

$$\overline{\Sigma_0(F_0, \dots, F_l)} = \overline{Z(F_0, \dots, F_l)},$$

and, in particular, $\Sigma_0(F_0, \dots, F_l)$ is a hypersurface.

Since the proof of these facts is the same as for Propositions 4.2 and 4.3, with the exception of more complicated notation coming from $l > 0$, we omit it.

5.2 Newton polyhedra of discriminants of complete intersections.

The study of the Newton polyhedron and leading coefficients of the discriminant $D_{F_0, \dots, F_l}^{\text{red}}$ can be reduced to the case $l = 0$ (which is studied in the previous section) by the Cayley trick, which represents $D_{F_0, \dots, F_l}^{\text{red}}$ as a product of discriminants of the form $D_{G_J}^{\text{red}}$ for linear combinations $G_J = \sum_{j \in J} \lambda_j F_j$ with indeterminate coefficients λ_j , where J runs over certain subsets of $\{0, \dots, l\}$.

More precisely, define the function G_J on $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k \times (\mathbb{C} \setminus 0)^{l+1}$ as $\sum_{j \in J} \lambda_j F_j$, where $\lambda_0, \dots, \lambda_l$ are coordinates on $(\mathbb{C} \setminus 0)^{l+1}$. Let e_0, \dots, e_l be the standard basis in \mathbb{Z}^{l+1} , denote the set $\bigcup_{j \in J} A_j \times \{e_j\}$ by $A_J \subset \mathbb{Z}^k \oplus \mathbb{Z}^{l+1}$. Then, for every $x \in \mathbb{T}^\tau$, the polynomial $G_J(x, \cdot)$ on $(\mathbb{C} \setminus 0)^k \times (\mathbb{C} \setminus 0)^{l+1}$ is contained in $\mathbb{C}[A_J]$, and the discriminant $D_{G_J}^{red}$ is defined by the formula $D_{G_J}^{red}(x) = D_{A_J}^{red}(G_J(x, \cdot))$.

These discriminants are related to the desired one as follows. For every $J \subset \{0, \dots, l\}$, denote the difference $\dim \sum_{j \in J} A_j - |J|$ by $\text{codim } J$.

THEOREM 5.4 (Cayley trick). *The discriminant $D_{F_0, \dots, F_l}^{red}$, $0 < l < k$, equals the product of the discriminants $D_{G_J}^{red}$ over all subsets $J \subset \{0, \dots, l\}$, such that $\text{codim } J \leq \text{codim } J'$ for every $J' \supset J$.*

This is Theorem 2.31 in the new notation. To describe the Newton polyhedra and leading coefficients of the discriminants $D_{G_J}^{red}$, and therefore those of $D_{F_0, \dots, F_l}^{red}$, we can apply Theorem 4.10 and Proposition 4.11 to the functions G_J under an appropriate condition of general position for their leading coefficients (see Definition 3.6). In many cases, the result of this computation can be written as an explicit formula for the Newton polyhedron of $D_{F_0, \dots, F_l}^{red}$; see, for example, Theorem 5.10 below. The aforementioned condition of general position can be formulated in terms of leading coefficients of the functions F_0, \dots, F_l as follows.

We denote the Newton polyhedron of $f_{a,i}$ by $\Delta_{a,i}$, and define the Newton polyhedron Δ_i of the function F_i as the convex hull of the set $\bigcup_{a \in A_i} \Delta_{a,i} \times \{a\} \subset \mathbb{R}^n \oplus \mathbb{R}^k$; then $F_i(z)$ can be represented as a power series $\sum_{b \in \Delta_i} c_{b,i} z^b$ for $z \in (\mathbb{C} \setminus 0)^n \times (\mathbb{C} \setminus 0)^k$. If Γ is a face of Δ_i , then we denote the function $\sum_{b \in \Gamma} c_{b,i} z^b$ by F_i^Γ .

DEFINITION 5.5. The leading coefficients of the functions $f_{a,i}$, $a \in A_i$, are said to be *in general position*, if, for every sequence $0 \leq i_1 < \dots < i_q \leq l$ and every collection of compatible bounded faces $\Gamma_{i_j} \subset \Delta_{i_j}$ (see Definition 1.13), such that the restriction of the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$ to $\Gamma_{i_1} + \dots + \Gamma_{i_q}$ is injective, $(0, \dots, 0)$ is a regular value of the polynomial map $(F_{i_1}^{\Gamma_{i_1}}, \dots, F_{i_q}^{\Gamma_{i_q}}) : (\mathbb{C} \setminus 0)^{n+k} \rightarrow \mathbb{C}^q$.

5.3 The case of analogous Newton polyhedra.

Under some additional assumptions on the sets A_0, \dots, A_l , the Cayley trick allows to explicitly compute the Newton polyhedron of the discriminant of a complete intersection as follows.

DEFINITION 5.6. Let A' be a face of a finite set $A \subset \mathbb{R}^k$, and let L be the vector subspace in \mathbb{R}^k , parallel to the affine span of A' . The A' -link of A is a (non-convex) polyhedron $\tilde{A} \setminus \tilde{A}' \subset \mathbb{R}^k/L$, where \tilde{A} and \tilde{A}' are the convex hulls of the images of the sets A and $A \setminus A'$ under the projection $\mathbb{R}^k \rightarrow \mathbb{R}^k/L$.

DEFINITION 5.7. Finite sets A and B in \mathbb{R}^k are said to be *analogous*, if there is a one-to-one correspondence between the posets of their faces, such that, for every pair of corresponding faces $A' \subset A$ and $B' \subset B$, the A' -link of A equals

the B' -link of B up to a parallel translation (in particular, the affine spans of A' and B' are parallel to the same subspace $L \subset \mathbb{R}^k$).

EXAMPLE 5.8. 1) If $A = B$, then A and B are analogous.
 2) If P and Q are analogous integer polyhedra (i.e. their dual fans coincide) and $k \in \mathbb{Z}$ is large enough, then the sets of integer points in kP and kQ are analogous. Note that those sets are not necessary analogous for $k = 1$. For example, the two sets on the picture in Subsection 1.5 have different links of their vertical faces.

Recall that the standard basis in \mathbb{Z}^{l+1} is denoted by e_0, \dots, e_l .

LEMMA 5.9. *If A_0, \dots, A_l in \mathbb{Z}^k are analogous, then, for every collection of corresponding faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$,*

- 1) $e^{A'_0, A_0} = \dots = e^{A'_l, A_l}$,
- 2) $A' = \bigcup_{i=0}^{l'} A'_i \times \{e_i\}$ is a face of $A = \bigcup_{i=0}^l A_i \times \{e_i\}$, and $e^{A', A} = e^{A'_0, A_0}$.

PROOF. Part 1 and Part 2 for $l' = l$ follow by the fact, that the Euler obstruction $e^{B', B}$ depends on the B' -link of B only (by definition, see Subsection 1.5). Since $e^{B', B}$ is the Euler obstruction of the B -toric variety at a point of its B' -orbit (Theorem 1.27), and since Euler obstruction is a local topological invariant, then $e^{A', A}$ does not depend on l' , and it is enough to prove Part 2 for $l' = l$. \square

In the notation of Subsection 5.1, let M be the lattice, generated by pairwise differences of points of the set $A_0 + \dots + A_l$, and let $\Delta_0, \dots, \Delta_l$ be the Newton polyhedra of the functions F_0, \dots, F_l . Recall that we denote the mixed fiber polyhedron of polyhedra P_1, \dots, P_q by the monomial $P_1 \cdot \dots \cdot P_q$.

THEOREM 5.10. 1) *If the leading coefficients of the functions $f_{a,i}$, $a \in A_i$, are in general position in the sense of Definition 5.5, the sets A_0, \dots, A_l are analogous and not contained in an affine hyperplane, then the Newton polyhedron of the discriminant $D_{F_0, \dots, F_l}^{red}$ equals*

$$\mathcal{N} = \frac{1}{|\mathbb{Z}^k/M|} \sum_{A'_0, \dots, A'_l} e^{A'_0, A_0} \sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = \dim A'_0 + 1}} \Delta_0(A'_0)^{a_0} \cdot \dots \cdot \Delta_l(A'_l)^{a_l},$$

where the collection (A'_0, \dots, A'_l) runs over all collections of corresponding faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$, including $A'_0 = A_0, \dots, A'_l = A_l$.

2) *If the leading coefficients of the functions $f_{a,i}$ are arbitrary, then the Newton polyhedron of the discriminant $D_{F_0, \dots, F_l}^{red}$ is contained in \mathcal{N} .*

An example of application is given in Subsection 5.6.

PROOF. By Theorem 5.4, we have $D_{F_0, \dots, F_l}^{red} = D_{F_{\{0, \dots, l\}}}^{red}$. Thus, the Newton polyhedron of $D_{F_0, \dots, F_l}^{red}$ is $|\mathbb{Z}^k/M|$ times smaller than the Newton polyhedron of $D_{F_{\{0, \dots, l\}}}$. We compute the latter one by Theorem 4.10, and simplify the answer by Lemmas 5.9 and 1.12(2). \square

5.4 The case of branched coverings and higher additivity.

The assumptions of Theorem 5.10 can be significantly relaxed, especially for large l . We illustrate this for $l = k$ (elimination theory) and for $l = k - 1$ (the projection of $F_0 = \dots = F_l = 0$ onto \mathbb{T}^r is typically a branched covering in this case).

Let $\sigma_m(t_1, \dots, t_l)$ be the symmetric function $\sum_{a_1 > 0, \dots, a_l > 0} t_1^{a_1} \dots t_l^{a_l}$ of formal variables.

LEMMA 5.11 (Higher additivity).

$$\begin{aligned} \sigma_m(t_0 + \tilde{t}_0, t_1, \dots, t_l) &= \sum_{\mu=1}^{\infty} \sigma_m(\underbrace{t_0, \dots, t_0}_{\mu}, \underbrace{\tilde{t}_0, \dots, \tilde{t}_0}_{\mu-1}, t_1, \dots, t_l) + \\ &+ \sigma_m(\underbrace{t_0, \dots, t_0}_{\mu-1}, \underbrace{\tilde{t}_0, \dots, \tilde{t}_0}_{\mu}, t_1, \dots, t_l) + 2\sigma_m(\underbrace{t_0, \dots, t_0}_{\mu}, \underbrace{\tilde{t}_0, \dots, \tilde{t}_0}_{\mu}, t_1, \dots, t_l). \end{aligned}$$

Note that there are only finitely many non-zero terms (those for $2\mu + k \leq m$) in the right hand side. For $m = l + 1$, the identity degenerates to $(t_0 + \tilde{t}_0)t_1 \dots t_l = t_0 t_1 \dots t_l + \tilde{t}_0 t_1 \dots t_l$. The proof is standard.

Let $\mathcal{M}_{\tau^\vee}(A_0)$ be the semigroup of all pairs of the form (A, Δ) , such that the finite set $A \subset \mathbb{Z}^k$ is analogous to $A_0 \subset \mathbb{Z}^k$, the polyhedron Δ is in \mathcal{M}_{τ^\vee} , and its image under the projection $\mathbb{R}^n \oplus \mathbb{R}^k \rightarrow \mathbb{R}^k$ equals the convex hull of A (this is a semigroup with respect to Minkowski addition of finite sets and polyhedra).

DEFINITION 5.12. The *higher mixed fiber polyhedron* is the collection of symmetric functions

$$\text{HP} : \underbrace{\mathcal{M}_{\tau^\vee}(A_0) \times \dots \times \mathcal{M}_{\tau^\vee}(A_0)}_{l+1} \rightarrow \mathcal{M}_{\tau^\vee}(0)$$

for $l \geq 0$, such that

$$1) \text{ HP}\left(\underbrace{(A, \Delta), \dots, (A, \Delta)}_{l+1}\right) = \sum_{A' \subset A} e^{A', A} \binom{\dim A'}{l} \int \Delta(A'),$$

for every $(A, \Delta) \in \mathcal{M}_{\tau^\vee}(A_0)$ with $\dim A = k$ (A' runs over all faces of A of dimension l or greater), and $\text{HP}\left((A, \Delta), \dots, (A, \Delta)\right) = \tau^\vee$ for $\dim A < k$;

$$\begin{aligned} 2) \text{ HP}(t_0 + \tilde{t}_0, t_1, \dots, t_l) &= \sum_{\mu=1}^{\infty} \text{HP}(\underbrace{t_0, \dots, t_0}_{\mu}, \underbrace{\tilde{t}_0, \dots, \tilde{t}_0}_{\mu-1}, t_1, \dots, t_l) + \\ &+ \text{HP}(\underbrace{t_0, \dots, t_0}_{\mu-1}, \underbrace{\tilde{t}_0, \dots, \tilde{t}_0}_{\mu}, t_1, \dots, t_l) + 2 \text{HP}(\underbrace{t_0, \dots, t_0}_{\mu}, \underbrace{\tilde{t}_0, \dots, \tilde{t}_0}_{\mu}, t_1, \dots, t_l) \end{aligned}$$

for all pairs $t_0, \tilde{t}_0, t_1, \dots, t_l$ in $\mathcal{M}_{\tau^\vee}(A_0)$.

By induction on $k - l$, these conditions uniquely define the function HP (at the base of the induction, for $l = k$, we have the definition of the mixed fiber polyhedron). On the other hand, by the lemma stated above, the polyhedron $\text{HP}\left((A_0, \Delta_0), \dots, (A_l, \Delta_l)\right) =$

$$= \sum_{A'_0, \dots, A'_l} e^{A'_0, A_0} \sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = \dim A'_0 + 1}} \Delta_0(A'_0)^{a_0} \cdot \dots \cdot \Delta_l(A'_l)^{a_l},$$

with (A'_0, \dots, A'_l) running over all collections of corresponding faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$, satisfies Definition 5.12. In particular, $\text{HP} = 0$ for $l > k$. We can now formulate Theorem 5.10 as follows.

THEOREM 5.13. *If A_0, \dots, A_l are analogous finite sets in \mathbb{R}^k , and $\sum_i A_i \times \{1\}$ generates $\mathbb{Z}^k \oplus \mathbb{Z}^1$, then, in the notation of Subsection 5.1, the Newton polyhedron of the discriminant $D_{F_0, \dots, F_l}^{\text{red}}$ equals $\text{HP}\left((A_0, \Delta_0), \dots, (A_l, \Delta_l)\right)$.*

Unexpectedly, as soon as we formulate Theorem 5.10 in this form, it can be generalized to non-analogous collections A_0, \dots, A_l in some cases (examples are Propositions 5.14 and 5.16 below), which motivates the following question:

To what extent one can relax the assumption that the arguments of HP are analogous in Definition 5.12, so that the higher mixed fiber polyhedron still exists and Theorem 5.13 remains valid?

In Subsection 5.2, we computed the Newton polyhedron of the discriminant $D_{F_0, \dots, F_l}^{\text{red}}$ of functions F_0, \dots, F_l , whose leading coefficients are in general position. We denote this Newton polyhedron by $\mathcal{N}_{\Delta_0, \dots, \Delta_l}^{A_0, \dots, A_l}$, where $\Delta_0, \dots, \Delta_l$ are the Newton polyhedra of the functions F_0, \dots, F_l . The following description of $\mathcal{N}_{\Delta_0, \dots, \Delta_k}^{A_0, \dots, A_k}$ is equivalent to Theorem 5.10 for $l = k$ and analogous sets A_0, \dots, A_k , but is valid for arbitrary sets A_0, \dots, A_k .

PROPOSITION 5.14. *Suppose that $l = k$, and the lattice \mathbb{Z}^k is generated by pairwise differences of elements of A_i for every $i = 0, 0', 1, \dots, k$. Then*
Additivity: $\mathcal{N}_{\Delta_0 + \Delta_{0'}, \Delta_1, \dots, \Delta_k}^{A_0 + A_{0'}, A_1, \dots, A_k} = \mathcal{N}_{\Delta_0, \dots, \Delta_k}^{A_0, \dots, A_k} + \mathcal{N}_{\Delta_{0'}, \Delta_1, \dots, \Delta_k}^{A_{0'}, A_1, \dots, A_k}$.
Unmixed case: If $\Delta_0 = \dots = \Delta_k$, and A_0 is not contained in a hyperplane, then $\mathcal{N}_{\Delta_0, \dots, \Delta_k}^{A_0, \dots, A_k} = \int \Delta_0$.

This is just another formulation of Theorem 3.3(1). We generalize this proposition to the case $l = k - 1$ as follows.

DEFINITION 5.15. Finite sets $A \subset \mathbb{Z}^k$ and $V \subset (\mathbb{Z}^k)^*$ are said to be *compatible*, if

- 1) V contains the primitive external normal covector to every codimension 1 face of the convex hull of A ,
- 2) for every linear function $v \in V$, the maximal and the next to the maximal values of v on A differ by 1,
- 3) pairwise differences of elements of A generate \mathbb{Z}^k .

Nota that, if A_1 and A_2 are compatible with the same V , it does not imply that A_1 and A_2 are analogous: the simplest example is $A_1 = \{(0, 0), (0, 1), (1, 0)\}$ and $A_2 = \{(0, 0), (0, -1), (-1, 0)\}$.

PROPOSITION 5.16. *Suppose that $l = k - 1$, and all the sets $\sum_{i \in I} A_i$, $I \subset \{0, 0', 1, \dots, k - 1\}$, are compatible with the same set $V \in (\mathbb{Z}^k)^*$. Then higher additivity:*

$$\mathcal{N}_{\Delta_0 + \Delta_{0'}, \Delta_1, \dots, \Delta_{k-1}}^{A_0 + A_{0'}, A_1, \dots, A_{k-1}} = \mathcal{N}_{\Delta_0, \dots, \Delta_{k-1}}^{A_0, \dots, A_{k-1}} + \mathcal{N}_{\Delta_{0'}, \Delta_1, \dots, \Delta_{k-1}}^{A_{0'}, A_1, \dots, A_{k-1}} + 2\mathcal{N}_{\Delta_0, \Delta_{0'}, \Delta_1, \dots, \Delta_{k-1}}^{A_0, A_{0'}, A_1, \dots, A_{k-1}},$$

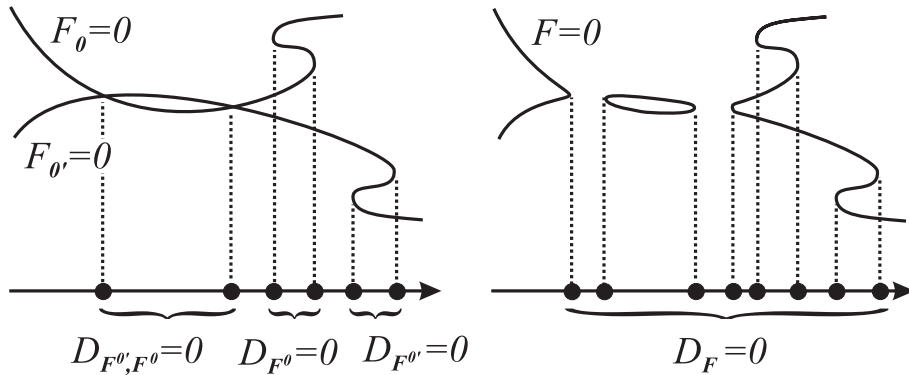
unmixed case: If $\Delta_0 = \dots = \Delta_{k-1}$, and A_0 has codimension 0, then

$$\mathcal{N}_{\Delta_0, \dots, \Delta_{k-1}}^{A_0, \dots, A_{k-1}} = k \int \Delta_0 - \sum_{A'} \int \Delta_0(A'),$$

where A' runs over all codimension 1 faces of A_0 .

In the same way as for $l = k$, these two identities are enough to compute $\mathcal{N}_{\Delta_0, \dots, \Delta_{k-1}}^{A_0, \dots, A_{k-1}}$. The proof of Proposition 5.16 follows the same lines as for Theorem 5.10; the computations are not affected by the fact that the sets A_0, \dots, A_{k-1} may be not analogous under the assumptions above.

REMARK. Another way to prove additivity in Proposition 5.16 is to consider functions F_i with generic leading coefficients and Newton polyhedra Δ_i for $i = 0, 0', 1, \dots, k - 1$, and a function F with generic leading coefficients and the Newton polyhedron $\Delta_0 + \Delta_{0'}$. Then, as F tends to the product $F_0 F_{0'}$, the discriminant $D_{F, F_1, \dots, F_{k-1}}$ tends to the product $D_{F_0, F_1, \dots, F_{k-1}} D_{F_{0'}, F_1, \dots, F_{k-1}} (D_{F_0, F_{0'}, F_1, \dots, F_{k-1}})^2$, as the following picture illustrates for $l = k - 1 = 0$.



We omit a detailed proof of Proposition 5.16, because its only purpose is to provide a motivation for the question, formulated after Theorem 5.13, but neither of the two mentioned ideas of the proof seem relevant to answer this question.

5.5 Bifurcation sets and their Newton polyhedra.

In this subsection we study one more problem, similar to the one studied in the first part of this section. Namely, in the notation of Section 5.1, we study the

minimal (closed) subset $S_{F_0, \dots, F_l} \subset \mathbb{T}^\tau$, such that the restriction of the projection $\mathbb{T}^\tau \times (\mathbb{C} \setminus 0)^k \rightarrow \mathbb{T}^\tau$ to $\{F_0 = \dots = F_l = 0\}$ is a fiber bundle outside of S_{F_0, \dots, F_l} . It is called the *bifurcation set* of the projection. Note that, in contrast to $\{D_{F_0, \dots, F_l}^{\text{red}} = 0\}$, the set S_{F_0, \dots, F_l} takes into account “singularities at infinity” of fibers of the projection $\{F_0 = \dots = F_l = 0\} \rightarrow \mathbb{T}^\tau$. For details and examples, see Subsection 2.4 where the universal case of this problem is studied.

We are interested in the Newton polyhedron of the equation of S_{F_0, \dots, F_l} , under the assumption that the Newton polyhedra of F_0, \dots, F_l are given, and their leading coefficients are in general position. If $\Delta_0, \dots, \Delta_l$ are the Newton polyhedra of the functions F_0, \dots, F_l , and the leading coefficients of these functions are in general position, then we denote the Newton polyhedron of the discriminant $D_{\lambda_0 F_0 + \dots + \lambda_l F_l}^{\text{red}}$ by $\mathcal{N}_{\Delta_0 * \dots * \Delta_l}^{A_0 * \dots * A_l}$ (see Theorem 4.10 for its computation).

THEOREM 5.17. *If the collection A_0, \dots, A_l is B -nondegenerate (see Subsection 2.4), and the leading coefficients of the functions $f_{a,i}$, $a \in A_i$, are in general position in the sense of Definition 5.5, then*

- 1) *the bifurcation set S_{F_0, \dots, F_l} is a hypersurface.*
- 2) *Assigning appropriate positive multiplicities to the components of the hypersurface $S_{F_0, \dots, F_l} \subset \mathbb{T}^\tau$ outside the maximal torus, it becomes a Cartier divisor, and the Newton polyhedron of its equation equals*

$$\sum_{A'_{j_1}, \dots, A'_{j_p}} \mathcal{N}_{\Delta(A'_{j_1}) * \dots * \Delta(A'_{j_p})}^{A'_{j_1} * \dots * A'_{j_p}},$$

where $A'_{j_1} \subset A_{j_1}, \dots, A'_{j_p} \subset A_{j_p}$ runs over all collections of compatible faces that can be extended to a collection of compatible faces $A'_0 \subset A_0, \dots, A'_l \subset A_l$ such that $\dim \sum_{j \in J} A'_j - \dim \sum_i A'_i \geq |J| - p$ for every $J \supset \{j_1, \dots, j_p\}$.

Note that the first assumption in this statement can be omitted, if Conjecture 2.28 is valid.

The proof of Theorem 5.17 is based on the following idea: define the function H_{F_0, \dots, F_l} on \mathbb{T}^τ as $H_{F_0, \dots, F_l}(x) = B_{A_0, \dots, A_l}(F_0(x, \cdot), \dots, F_l(x, \cdot))$ (see Subsection 2.4 for the definition of the discriminant B_{A_0, \dots, A_l}). Then $\{H_{F_0, \dots, F_l} = 0\}$ is the equation of S_{F_0, \dots, F_l} (the proof follows the same lines as the proof of Propositions 4.2 and 4.3), and can be expressed in terms of discriminants by Corollary 2.32.

COROLLARY 5.18. *If each of A_0, \dots, A_l is the set of integer lattice points in a Delzant polytope, these Delzant polytopes have the same dual fan, and the leading coefficients of the functions $f_{a,i}$, $a \in A_i$, are in general position in the sense of Definition 5.5, then $e^{A'_i, A_i} = (-1)^{\dim A_i - \dim A'_i}$ for every face A'_i , and the Newton polyhedron of the equation of S_{F_0, \dots, F_l} equals*

$$\sum_{A'_0, \dots, A'_l} \sum_{\substack{a_0 > 0, \dots, a_l > 0 \\ a_0 + \dots + a_l = k+1}} \Delta_0(A'_0)^{a_0} \cdot \dots \cdot \Delta_l(A'_l)^{a_l},$$

where A'_0, \dots, A'_l runs over all collections of compatible faces of the sets A_0, \dots, A_l , including $A'_0 = A_0, \dots, A'_l = A_l$.

This is a corollary of Theorem 5.17, Theorem 5.10 and Proposition 2.29.

5.6 Example and computability questions.

EXAMPLE. Consider the first coordinate in the torus $(\mathbb{C} \setminus 0)^3$ as the “height function”, and the first coordinate line in \mathbb{R}^3 as the “vertical” line. We will compute the number of critical points of the restriction of the height function to the curve $f = g = 0$ and to the surface $f = 0$ for generic equations f and g with a given Newton polyhedron $\Delta \subset \mathbb{R}^3$. We can compute these numbers by the global version of Theorems 5.10 and 4.10 respectively with $k = 2$ and $n = 1$, because the number of critical points of the height function is the degree of the discriminant of the projection onto the vertical coordinate line. The answer is as follows.

Let S_1, \dots, S_n be the areas of vertical faces of Δ , let l_1, \dots, l_n be the lengths of its vertical edges, and let $d_1, \dots, d_n, e_1, \dots, e_n$ be the Euler obstructions of Δ at these faces and edges. In this notation, the number of critical points of the restriction of the height function to the curve $f = g = 0$ equals

$$12 \operatorname{Vol} \Delta + 2 \sum_i d_i S_i.$$

The number of critical points of the restriction of the height function to the surface $f = 0$ equals

$$6 \operatorname{Vol} \Delta + 2 \sum_i d_i S_i + \sum_i e_i l_i.$$

To explain the first coefficient in the first of these answers informally, note that the desired critical points are solutions to the following system of equations:

$$f = g = \det \begin{pmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{pmatrix} = 0.$$

The Newton polyhedron of the first two equations is denoted by Δ , thus the Newton polyhedron of the last equation “approximately” equals 2Δ . Thus, if the Kouchnirenko-Bernstein formula were applicable to this system of the equations, then it would have approximately $6 \operatorname{MV}(\Delta, \Delta, 2\Delta) = 12 \operatorname{Vol}(\Delta)$ solutions. Although this illustrates why the coefficient in the first answer equals 12, neither the Newton polyhedron of the last solution equals 2Δ in general, nor are the equations generic with respect to their Newton polyhedra. Thus, such a straightforward way to count critical points would be irrelevant.

COMPUTABILITY QUESTIONS. Since Euler obstructions of polyhedra may be negative, many statements and computations above involve subtraction of polyhedra. The difference of polyhedra A and B is by definition the solution of the equation $B + X = A$. It does not always exist (e.g. the difference of polygons P and Q exists if and only if, for every edge of Q , we can find a longer or equal edge of P with the same external normal). If the difference $A - B$ exists, then it is unique by the following reason.

Recall that the *support function* $A(\cdot) : (\mathbb{R}^m)^* \rightarrow \mathbb{R} \sqcup \{-\infty\}$ of a polyhedron $A \subset \mathbb{R}^m$ is defined by the equality $A(l) = \min_{a \in A} l(a)$. If the difference of support functions $A(\cdot) - B(\cdot)$ is not concave, then the Minkowski difference $A - B$ does

not exist, otherwise $A - B$ can be reconstructed from its support function, which equals $A(\cdot) - B(\cdot)$.

Thus, when computing Minkowski linear combinations of mixed fiber polyhedra that appear throughout the paper, it is reasonable to encode polyhedra with their support functions. Then Minkowski summation and subtraction is substituted with summation and subtraction of support functions, and mixed fiber polyhedra can be computed by means of [SY] (where the corner locus of the support function of the mixed fiber polyhedron $\text{MP}(P_0, \dots, P_k)$ is computed in terms of the corner loci of the support functions of the arguments P_0, \dots, P_k), or by means of Proposition 1.10 (in Appendix, this form of the answer is also represented as the mixed volume of certain bounded virtual polyhedra, i.e. the tropical intersection number of the corner loci of their support functions).

For instance, denote the *support face* of the polyhedron $P \subset \mathbb{R}^m$, at which a linear function $\mu \in (\mathbb{R}^m)^*$ attains its minimum, by P^μ , and discuss the following problem regarding the Newton polyhedron $\mathcal{N} \subset \mathbb{R}^n$ of the discriminant that was discussed in the introduction:

Given the Newton polyhedra $\Delta_0, \dots, \Delta_l$ and a linear function $\mu \in (\mathbb{R}^n)^*$ that attains its minimum at a vertex of \mathcal{N} , compute the coordinates of the vertex \mathcal{N}^μ .

We restrict our attention to a coordinate function that we denote by λ , and compute the coordinate $\mathcal{N}^\mu(\lambda)$ as follows:

- 1) Since $(P \pm Q)^\mu(\lambda) = P^\mu(\lambda) \pm Q^\mu(\lambda)$, the theorem in the introduction represents $\mathcal{N}^\mu(\lambda)$ as a linear combination of values $R^\mu(\lambda)$, where R runs over mixed fiber polyhedra of the form $\text{MP}(\Delta_{i_0}, \dots, \Delta_{i_k})$;
- 2) Since the support function of the mixed fiber polyhedron $\text{MP}(P_0, \dots, P_k) \subset \mathbb{R}^n$ for a linear function $\mu \in (\mathbb{R}^n)^*$ equals the Minkowski sum of mixed fiber polyhedra $\text{MP}(P_0^{\mu'}, \dots, P_k^{\mu'})$ over all μ' whose restriction to \mathbb{R}^n equals μ (see [McM] or [EKh]), we can represent $\text{MP}(\Delta_{i_0}, \dots, \Delta_{i_k})^\mu(\lambda)$ as the sum of $\text{MP}(\Delta_{i_0}^{\mu'}, \dots, \Delta_{i_k}^{\mu'})^\mu(\lambda)$;
- 3) The latter value can be computed by Proposition 1.10.

In [DFS], the same problem (of finding the support vertex of a given linear function) for the Newton polyhedron of the discriminant D_A is solved in another way, which has two advantages: it is positive (i.e. the algorithm is based on formulas that do not involve subtraction) and it will work for tropicalizations of discriminant sets of higher codimension (i.e. when A is dual defect). Thus, it would be useful to generalize the technique of [DFS] to our setting.

6 Appendix ([E08]): Mixed fiber bodies.

The notion of the mixed fiber polytope is a natural generalization of the mixed volume and the Minkowski integral (see [BS] or Definition 6.1). It is closely related to elimination theory, see [EKh], [ST], or Theorem 3.3. The existence of mixed fiber polytopes was predicted in [McD] and proved in [McM]. One can extend the notion of the mixed fiber polytope to convex bodies by continuity. We present a direct proof of the existence of mixed fiber bodies, which does not exploit the reduction to polytopes by continuity (see the proof of Theorem 6.2).

It is based on an explicit formula 6.1.(*) for the support function of a mixed fiber body. Other applications of this formula include the proof of Theorem 3.3 and a certain monotonicity property for mixed fiber bodies (Theorem 6.30). For simplicity, we discuss bounded convex bodies here, although this restriction can be easily omitted (see Section 1.2 for unbounded mixed fiber polyhedra).

6.1 Mixed fiber bodies

Let L and M be real vector spaces of dimension l and m respectively, and let μ be a volume form on M . Denote the projections of $L \oplus M$ to L and M by u and v respectively. Let $\Delta \subset L \oplus M$ be a convex body, i. e. a compact set, which contains all the line segments connecting any pair of its points. For a convex body and a point $a \in M$, denote the *fiber* $u(\Delta \cap v^{(-1)}(a))$ of Δ by Δ_a . Recall that the *support function* $B(\cdot) : L^* \rightarrow \mathbb{R}$ of a convex body $B \subset L$ is defined as $B(\gamma) = \max_{b \in B} \langle \gamma, b \rangle$ for every covector $\gamma \in L^*$.

DEFINITION 6.1. For a convex body $\Delta \subset L \oplus M$, its *Minkowski integral* is the convex body $B \subset L$, such that its support function equals the integral of the support functions of the fibers Δ_a , where a runs over $v(\Delta)$:

$$B(\gamma) = \int_{v(\Delta)} \Delta_a(\gamma) \mu \text{ for every } \gamma \in L^*.$$

The Minkowski integral is denoted by $\int \Delta \mu$.

This definition is slightly different from the original one (see [BS]). We discuss this difference in Section 6.4.

Denote the set of all convex bodies in a real vector space K by $\mathcal{C}(K)$. This set is a semigroup with respect to the Minkowski summation $A + B = \{a + b \mid a \in A, b \in B\}$.

THEOREM 6.2. *There exists a unique symmetric Minkowski-multilinear map $\text{MP}_\mu : \underbrace{\mathcal{C}(L \oplus M) \times \dots \times \mathcal{C}(L \oplus M)}_{m+1} \rightarrow \mathcal{C}(L)$, such that*

$$\text{MP}_\mu(\Delta, \dots, \Delta) = \int \Delta \mu \text{ for every convex body } \Delta \subset L \oplus M.$$

DEFINITION 6.3. The convex body $\text{MP}_\mu(\Delta_0, \dots, \Delta_m)$ is called the *mixed fiber body* of bodies $\Delta_0, \dots, \Delta_m$.

It is quite easy to see that the Minkowski integral is a homogeneous polynomial, and, thus, admits such a polarization in the class of virtual convex bodies (see Definition 6.9). The fact that this polarization gives actual convex bodies rather than virtual convex bodies is the most important part of the assertion.

PROOF OF THEOREM 6.2.

DEFINITION 6.4. The *shadow volume* $S_\mu(B)$ of a convex body $B \subset \mathbb{R} \oplus M$ is the integral $\int_{B_0} \varphi \mu$, where B_0 is the projection of B to M , and φ is the maximal function on B_0 such that its graph is contained in B (in other words, $\varphi(a) = \max_{(t,a) \in B} t$ for every $a \in B_0$).

One can reformulate the definition of the Minkowski integral as follows. For a covector $\gamma \in L^*$ and a convex body $\Delta \subset L \oplus M$, denote the image of Δ under the projection $(\gamma, \text{id}) : L \oplus M \rightarrow \mathbb{R} \oplus M$ by $\Gamma_\Delta(\gamma)$.

LEMMA 6.5. *The value of the support function of the Minkowski integral $\int \Delta \mu$ at a covector $\gamma \in L^*$ equals the shadow volume of the body $\Gamma_\Delta(\gamma) \subset \mathbb{R} \oplus M$.*

This lemma implies that, instead of constructing mixed fiber bodies, it is enough to construct the mixed shadow volume in the following sense.

THEOREM 6.6. *There exists a unique symmetric Minkowski-multilinear function MS_μ of $m+1$ convex bodies in $\mathbb{R} \oplus M$ such that $\text{MS}_\mu(B, \dots, B) = S_\mu(B)$ for every convex body $B \subset \mathbb{R} \oplus M$.*

The proof is given in Section 6.2 and is based on an explicit formula for the function MS_μ , which is later used in the proof of Theorem 6.2 (see Lemma 6.26).

DEFINITION 6.7. The number $\text{MS}_\mu(B_0, \dots, B_m)$ is called the *mixed shadow volume* of convex bodies $B_0, \dots, B_m \subset \mathbb{R} \oplus M$.

Note that mixed shadow volume is a special case of mixed volume of pairs (see Proposition 6.27).

If the existence of mixed fiber bodies is proved, then Lemma 6.5 and Theorem 6.6 imply that the value of the support function of the mixed fiber body $\text{MP}_\mu(\Delta_0, \dots, \Delta_m)$ at a covector $\gamma \in L^*$ equals the mixed shadow volume $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$. We reverse this argument, using the following fact. Recall that a function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ is said to be *positively homogeneous*, if $f(ta) = tf(a)$ for all $a \in \mathbb{R}^k$ and $t \geq 0$.

THEOREM 6.8. *For any collection of convex bodies $\Delta_0, \dots, \Delta_m \subset L \oplus M$, the expression $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$ is a convex positively homogeneous function of a covector $\gamma \in L^*$.*

The proof is given in Section 6.3.

Theorem 6.8 implies that, for any collection of convex bodies $\Delta_0, \dots, \Delta_m \subset L \oplus M$, the expression $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$ defines the support function of some convex body $B \in L$:

$$\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma)) = B(\gamma) \text{ for every } \gamma \in L^*. \quad (*)$$

This body B satisfies the definition of the mixed convex body of $\Delta_0, \dots, \Delta_m$ by Lemma 6.5 and Theorem 6.6. \square

6.2 Mixed shadow volume. Proof of Theorem 6.6.

The shadow volume of a convex body $\Delta \subset \mathbb{R} \oplus M$ is equal to the volume of some virtual convex body $[\Delta]$ associated with Δ (see Definition 6.13 and Lemma 6.15 below). Since the correspondence $\Delta \rightarrow [\Delta]$ is Minkowski-linear, one can define the mixed shadow volume of convex bodies $\Delta_0, \dots, \Delta_m \subset \mathbb{R} \oplus M$ as the mixed volume of virtual bodies $[\Delta_0], \dots, [\Delta_m]$, which implies the existence of the mixed

shadow volume. To formulate this in detail, recall the definition of a virtual convex body.

The *Grothendieck group* Λ_G of a commutative semigroup Λ with the cancellation law $(a + c = b + c \Rightarrow a = b)$ is the group of formal differences of elements from Λ . In more detail, it is the quotient of the set $\Lambda \times \Lambda$ by the equivalence relation $(a, b) \sim (c, d) \Leftrightarrow a + d = b + c$, with operations $(a, b) + (c, d) = (a + c, b + d)$ and $-(a, b) = (b, a)$. The map, which carries every $a \in \Lambda$ to $(a + a, a) \in \Lambda_G$, is an inclusion $\Lambda \hookrightarrow \Lambda_G$. An element of the form $(a + a, a) \in \Lambda_G$ is said to be *proper* and is usually identified with $a \in \Lambda$. Under this convention, one can write $(a, b) = a - b$.

DEFINITION 6.9. *The group of virtual bodies in a real vector space K is the Grothendieck group of the semigroup of convex bodies in K (with respect to the operation of Minkowski summation).*

The classical operation of taking the mixed volume can be extended to virtual bodies by linearity. This extension is unique, but fails to be increasing (for example, $MV(-A, A) > MV(-A, 2A)$ for a convex polygon A).

DEFINITION 6.10. Let μ be a translation invariant volume form on a real vector space K of dimension n . The *mixed volume* MV_μ is the symmetric Minkowski-multilinear function of n virtual bodies in K , such that $MV_\mu(\Delta, \dots, \Delta)$ equals the volume of Δ in the sense of the form μ for every convex body $\Delta \subset K$.

DEFINITION 6.11. For the difference Δ of two convex bodies Δ_1 and Δ_2 in K , the *support function* $\Delta(\cdot) : K^* \rightarrow \mathbb{R}$ is $\Delta(\gamma) = \Delta_1(\gamma) - \Delta_2(\gamma)$.

One can reformulate the definition of the group of virtual bodies more explicitly as follows. A function $f : K \rightarrow \mathbb{R}$ is called a *DC function* if it can be represented as the difference of two convex functions.

LEMMA 6.12. *The map which carries every virtual body Δ to its support function $\Delta(\cdot)$ is an isomorphism between the group of virtual bodies in a real vector space K and the group of positively homogenous DC functions on the dual space K^* .*

PROOF OF THEOREM 6.6. Let M be an m -dimensional real vector space. Denote the ray $\{(t, 0) \mid t \leq 0\} \subset \mathbb{R} \oplus M$ by l_- , and denote the half-space $\{(t, x) \mid t \geq a\} \subset \mathbb{R} \oplus M$ by H_a .

DEFINITION 6.13. The *shadow* $[\Delta]$ of a convex body $\Delta \subset \mathbb{R} \oplus M$ is the difference of the convex bodies $(\Delta + l_-) \cap H_a$ and $l_- \cap H_a$, where a is a negative number such that $H_a \supset \Delta$.

This definition does not depend on the choice of a , because one can reformulate it in terms of support functions as follows. For every covector $\gamma = (t, \gamma_0) \in (\mathbb{R} \oplus M)^*$, denote the covector $(\max\{0, t\}, \gamma_0) \in (\mathbb{R} \oplus M)^*$ by $[\gamma]$. Then $[\Delta](\gamma) = \Delta([\gamma])$ for every γ .

Denote the unit volume form on \mathbb{R} by dt . The function which assigns the number $MV_{dt \wedge \mu}([\Delta_0], \dots, [\Delta_m])$ to every collection of convex bodies $\Delta_0, \dots, \Delta_m$

in $\mathbb{R} \oplus M$ is symmetric Minkowski-multilinear by Lemma 6.14 below and assigns the shadow volume $S_\mu(\Delta)$ to the collection (Δ, \dots, Δ) for every convex body Δ by Lemma 6.15 below. Thus, it satisfies the definition of the mixed shadow volume. Its uniqueness follows from Lemma 6.17 below. \square

LEMMA 6.14. $[\Delta_1 + \Delta_2] = [\Delta_1] + [\Delta_2]$.

PROOF. $[\Delta_1 + \Delta_2](\gamma) = \Delta_1([\gamma]) + \Delta_2([\gamma]) = [\Delta_1](\gamma) + [\Delta_2](\gamma)$ for every covector γ . \square

LEMMA 6.15. $S_\mu(\Delta) = \text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$.

PROOF. If $\Delta \subset H_0$ in the notation of Definition 6.13, then the shadow $[\Delta]$ equals the convex body $(\Delta + l_-) \cap H_0$ by definition, and both $S_\mu(\Delta)$ and $\text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$ equal the volume of $(\Delta + l_-) \cap H_0$. One can reduce the statement of Lemma 6.15 to this special case: substitute an arbitrary body Δ by a shifted body $\Delta + \{v\} \subset H_0$, where v is a vector of the form $(s, 0) \in \mathbb{R} \oplus M$, and note that both $S_\mu(\Delta)$ and $\text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$ increase by s times the volume of the projection of $\Delta \subset \mathbb{R} \oplus M$ to M . The latter fact follows by the definition for the shadow volume $S_\mu(\Delta)$, and follows from Lemma 6.16 for the mixed volume $\text{MV}_{dt \wedge \mu}([\Delta], \dots, [\Delta])$. \square

LEMMA 6.16. $\text{MV}_{dt \wedge \mu}([\Delta_0 + \{(s, 0)\}], [\Delta_1], \dots, [\Delta_m]) =$
 $= \text{MV}_{dt \wedge \mu}([\Delta_0], \dots, [\Delta_m]) + \frac{s}{m+1} \text{MV}_\mu(B_1, \dots, B_m),$

where the convex body $B_j \subset M$ is the projection of $\Delta_j \subset \mathbb{R} \oplus M$ to M and s is a positive number.

PROOF. $\text{MV}_{dt \wedge \mu}([\{(s, 0)\}], [\Delta_1], \dots, [\Delta_m]) = \frac{s}{m+1} \text{MV}_\mu(B_1, \dots, B_m)$, which is a corollary of the following well known formula (one can consider $N = \mathbb{R} \oplus M$ and $L = \mathbb{R} \times \{0\}$). Let A_1, \dots, A_n be convex bodies in an n -dimensional real vector space N , suppose that A_1, \dots, A_l are contained in an l -dimensional subspace $L \subset N$, and denote the projection $N \rightarrow N/L$ by p . Then $n! \text{MV}_{\mu \wedge \mu'}(A_1, \dots, A_n) = (n-l)! \text{MV}_\mu(pA_{l+1}, \dots, pA_n) \cdot l! \text{MV}_{\mu'}(A_1, \dots, A_l)$, where μ and μ' are volume forms on N/L and L . \square

LEMMA 6.17.

$$\text{MS}_\mu(\Delta_0, \dots, \Delta_m) = \frac{1}{(m+1)!} \sum_{0 \leq i_1 < \dots < i_p \leq m} (-1)^{m+1-p} S_\mu(\Delta_{i_1} + \dots + \Delta_{i_p}).$$

PROOF. To prove the identity

$$n!l(a_1, \dots, a_n) = \sum_{1 \leq i_1 < \dots < i_p \leq n} (-1)^{n-pl} (a_{i_1} + \dots + a_{i_p}, \dots, a_{i_1} + \dots + a_{i_p})$$

for every symmetric multilinear function l , open the brackets and collect like terms in the right hand side. \square

The following lemma describes how the mixed shadow volume changes under translation and dilatation of arguments along the line $\mathbb{R} \times \{0\} \subset \mathbb{R} \oplus M$.

LEMMA 6.18. 1) Let $D_s : \mathbb{R} \oplus M \rightarrow \mathbb{R} \oplus M$ be a dilatation along $\mathbb{R} \times \{0\}$, i. e. $D_s(t, x) = (st, x)$ for all $t \in \mathbb{R}$ and $x \in M$. Then $\text{MS}_\mu(D_s\Delta_0, \dots, D_s\Delta_m) = s \text{MS}_\mu(\Delta_0, \dots, \Delta_m)$ for every non-negative s .
 2) Let $T_s : \mathbb{R} \oplus M \rightarrow \mathbb{R} \oplus M$ be a translation, $T_s(t, x) = (t + s, x)$ for all $t \in \mathbb{R}$ and $x \in M$. Then $\text{MS}_\mu(T_s\Delta_0, \Delta_1, \dots, \Delta_m) = \text{MS}_\mu(\Delta_0, \dots, \Delta_m) + \frac{s}{m+1} \text{MV}_\mu(B_1, \dots, B_m)$, where the convex body $B_j \subset M$ is the projection of $\Delta_j \subset \mathbb{R} \oplus M$ to M .

In particular, the mixed shadow volume is not translation invariant.

PROOF. Part 1 follows from the definition of shadows and the equality $\text{MV}_{dt \wedge \mu}(D_s\Delta_0, \dots, D_s\Delta_m) = s \text{MV}_{dt \wedge \mu}(\Delta_0, \dots, \Delta_m)$ for convex bodies $\Delta_0, \dots, \Delta_m$. Part 2 follows Lemma 6.16. \square

6.3 Convexity of mixed shadow volume. Proof of Theorem 6.8.

DEFINITION 6.19. A set with a convexity structure is a pair (U, C) , where U is an arbitrary set and C is an arbitrary map $U \times (0, 1) \times U \rightarrow U$.

EXAMPLE 6.20. The pair $(\mathbb{R}^k, C_{\mathbb{R}^k})$, where $C_{\mathbb{R}^k}(u, t, v) = tu + (1 - t)v$, is a set with a convexity structure.

DEFINITION 6.21. Let (U, C) and (V, D) be two sets with convexity structures, and let \leq be a partial order on V . A map $f : U \rightarrow V$ is said to be a *convex map from (U, C) to (V, D, \leq)* , if $f(C(u, t, v)) \leq D(f(u), t, f(v))$ for all triples $(u, t, v) \in U \times (0, 1) \times U$.

EXAMPLE 6.22. For a map from $(\mathbb{R}^k, C_{\mathbb{R}^k})$ to $(\mathbb{R}^1, C_{\mathbb{R}^1}, \leq)$, this definition coincides with the classical one.

LEMMA 6.23 (tautological). *If maps $f : (U, C) \rightarrow (V, D, \leq)$ and $g : (V, D) \rightarrow (W, E, \leq)$ are convex, and g is increasing, then their composition is convex.*

We apply this lemma to prove Theorem 6.8 as follows: the map which assigns the mixed shadow volume $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$ to every covector $\gamma \in L^*$, and whose convexity we wish to prove, can be represented as a composition of simpler maps (see the diagram (**) below), whose convexity and monotonicity are almost obvious. The proof of their convexity and monotonicity occupies the rest of this subsection, and implies the convexity for the map of Theorem 6.8 by Lemma 6.23 (see the end of this subsection for details).

For a convex body $B \subset M$, let $\mathcal{C}(B)$ be the set of all convex bodies $\Delta \subset \mathbb{R} \oplus M$ such that the projection of Δ to M equals B . Introduce the shadow and the Minkowski convexity structures C_S and C_M on $\mathcal{C}(B)$ as follows. Consider convex bodies $\Delta_i = \{(t, a) \mid a \in B, t \in [\psi_i(a), \varphi_i(a)]\}$, $i = 1, 2$, in $\mathcal{C}(B)$, where φ_i and $-\psi_i$ are concave functions on B . Then, by definition,

$$C_S(\Delta_1, \alpha, \Delta_2) = \left\{ (t, a) \mid a \in B, t \in [\alpha\psi_1(a) + (1-\alpha)\psi_2(a), \alpha\varphi_1(a) + (1-\alpha)\varphi_2(a)] \right\},$$

$$C_M(\Delta_1, \alpha, \Delta_2) = \alpha\Delta_1 + (1 - \alpha)\Delta_2, \quad \Delta_1 \leq \Delta_2 \Leftrightarrow \varphi_1 \leq \varphi_2.$$

For convex bodies $\Delta_0, \dots, \Delta_m \subset L \oplus M$, denote the projection of Δ_j to M by B_j , and consider the maps

$$\begin{aligned} (L^*, C_{L^*}) &\xrightarrow{(\Gamma_{\Delta_0}, \dots, \Gamma_{\Delta_m})} (\mathcal{C}(B_0) \times \dots \times \mathcal{C}(B_m), C_S, \leq) \xrightarrow{(\text{id}, \dots, \text{id})} \\ &\rightarrow (\mathcal{C}(B_0) \times \dots \times \mathcal{C}(B_m), C_M, \leq) \xrightarrow{\text{MS}_\mu} (\mathbb{R}, C_{\mathbb{R}}, \leq), \end{aligned} \quad (**)$$

where the map id sends every convex body to itself.

LEMMA 6.24. *If a convex body $B \subset M$ is the projection of a convex body $\Delta \subset L \oplus M$, then the map $\Gamma_\Delta : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(B), C_S, \leq)$ is convex.*

PROOF. If $\dim M = 0$, then the body $\Gamma_\Delta(\gamma) \subset \mathbb{R} \oplus \{0\}$ is the segment $[-\Delta(-\gamma), \Delta(\gamma)]$ for every $\gamma \in L^*$, and the convexity of Γ_Δ follows from the convexity of the support function $\Delta(\cdot)$. One can reduce the general statement to this special case, because the convexity of the map $\Gamma_\Delta : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(B), C_S, \leq)$ is equivalent to the convexity of the maps $\Gamma_{\Delta_a} : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(\{0\}), C_S, \leq)$ for all points $a \in B$ (see the first paragraph of Appendix for the definition of the fiber Δ_a of the body Δ). \square

LEMMA 6.25. *If Δ_1 and Δ_2 are convex bodies from $\mathcal{C}(B)$, and $\alpha \in (0, 1)$, then $C_S(\Delta_1, \alpha, \Delta_2) \leq C_M(\Delta_1, \alpha, \Delta_2)$.*

PROOF. Every point (t, a) of the left hand side can be represented as $(\alpha s_1 + (1 - \alpha)s_2, a)$, where $(s_1, a) \in \Delta_1$ and $(s_2, a) \in \Delta_2$. Thus, it equals $\alpha(s_1, a) + (1 - \alpha)(s_2, a)$, which is contained in the right hand side. \square

LEMMA 6.26. *If $\Delta_j^0 \in \mathcal{C}(B_j)$, $\Delta_j^1 \in \mathcal{C}(B_j)$ and $\Delta_j^0 \leq \Delta_j^1$ for $j = 0, \dots, m$, then $\text{MS}_\mu(\Delta_0^0, \dots, \Delta_m^0) \leq \text{MS}_\mu(\Delta_0^1, \dots, \Delta_m^1)$.*

PROOF. Denote the convex body $\{0\} \times B_j \subset \mathbb{R} \oplus M$ by \tilde{B}_j , and denote the half-space $\mathbb{R}_{\geq 0} \times M$ by H . Shifting the bodies Δ_j^i and using part 2 of Lemma 6.18, we can assume without loss of generality, that $\Delta_j^i \subset H$ for all $i = 1, 2$ and $j = 0, \dots, m$. Under this assumption, the shadows $[\Delta_j^i]$ of bodies Δ_j^i (see Definition 6.13) are convex hulls $\text{conv}(\Delta_j^i \cup \tilde{B}_j)$. In particular, the shadows $[\Delta_j^i]$ are convex bodies, and $[\Delta_j^0] \subset [\Delta_j^1]$. Since the mixed shadow volume equals the mixed volume of shadows, the inequality $\text{MS}_\mu(\Delta_0^0, \dots, \Delta_m^0) \leq \text{MS}_\mu(\Delta_0^1, \dots, \Delta_m^1)$ follows from the monotonicity of the mixed volume of convex bodies. \square

PROOF OF THEOREM 6.8. Positive homogeneity follows from part 1 of Lemma 6.18. To prove convexity, represent the map which assigns the mixed shadow volume $\text{MS}_\mu(\Gamma_{\Delta_0}(\gamma), \dots, \Gamma_{\Delta_m}(\gamma))$ to every covector $\gamma \in L^*$, as a composition of simpler maps (**). These three maps are convex and increasing.

Namely, the convexity of the map $\Gamma_{\Delta_j} : (L^*, C_{L^*}) \rightarrow (\mathcal{C}(B_j), C_S, \leq)$ is proved in Lemma 6.24. The increasing monotonicity of $\text{id} : (\mathcal{C}(B_j), C_S, \leq) \rightarrow (\mathcal{C}, C_M, \leq)$ is tautological, the convexity follows from Lemma 6.25. The convexity of the mixed shadow volume follows from its linearity, and the increasing monotonicity of the mixed shadow volume is the statement of Lemma 6.26.

Since these maps are convex and increasing, their composition is also convex by Lemma 6.23. \square

6.4 Remarks.

MIXED VOLUMES OF PAIRS. Mixed shadow volume is a special case of mixed volume of pairs (see Definition 1.3). Let p be the projection $\mathbb{R} \oplus \mathbb{R}^k \rightarrow \{0\} \times \mathbb{R}^k$, and let l_- be the ray $\{(t, 0, \dots, 0) \mid t \leq 0\} \subset \mathbb{R} \oplus \mathbb{R}^k$. For a convex body $\Delta \subset \mathbb{R} \oplus \mathbb{R}^k$, denote the pair $(\Delta + l_-, p(\Delta) + l_-) \in \text{BP}_{l_-}$ by $\tilde{\Delta}$.

PROPOSITION 6.27. $\text{MS}_\mu(\Delta_0, \dots, \Delta_k) = \text{MV}(\tilde{\Delta}_0, \dots, \tilde{\Delta}_k)$ for every collection of convex bodies $\Delta_0, \dots, \Delta_k$ in $\mathbb{R} \oplus \mathbb{R}^k$.

PROOF. This equality follows by definitions if $\Delta_0 = \dots = \Delta_k$. The general statement follows from this special case by uniqueness of the mixed shadow volume. \square

BILLERA-STURMFELS VERSION OF MINKOWSKI INTEGRAL. The original definition of the fiber integral is slightly different from Definition 6.1. Let $p : N \rightarrow K$ be a projection of an n -dimensional real vector space to a k -dimensional one, and let μ be a volume form on K .

DEFINITION 6.28 ([BS]). For a convex body $\Delta \subset N$, the set of all points of the form $\int_{p(\Delta)} s \mu \in N$, where $s : p(\Delta) \rightarrow \Delta$ is a continuous section of the projection p , is called the *Minkowski integral* of Δ and is denoted by $\int^{BS} \Delta \mu$.

Definitions 6.1 and 6.28 are related as follows. If, combining notation from these definitions, we assume that $N = L \oplus M$ and p is the projection $L \oplus M \rightarrow M$, then the convex body $\int^{BS} \Delta \mu$ is contained in a fiber of p , and $\int \Delta \mu$ is the image of $\int^{BS} \Delta \mu$ under the projection $L \oplus M \rightarrow L$.

One can reduce Definition 6.28 to Definition 6.1 as well. This time, combining notation from these definitions, suppose that $L = N$, $M = K$, and the body Δ^{diag} consists of points $(a, p(a)) \in L \oplus M$, where a runs over all points of a convex body $\Delta \subset N$. Then $\int^{BS} \Delta \mu = \int \Delta^{diag} \mu$. In particular, one can denote $\text{MP}_\mu(\Delta_0^{diag}, \dots, \Delta_m^{diag})$ by $\text{MP}_\mu^{BS}(\Delta_0, \dots, \Delta_m)$ and reformulate Theorem 6.2 for the Billera-Sturmfels version of mixed fiber bodies.

THEOREM 6.29. *There exists a unique symmetric Minkowski-multilinear map $\text{MP}_\mu^{BS} : \underbrace{\mathcal{C}(N) \times \dots \times \mathcal{C}(N)}_{k+1} \rightarrow \mathcal{C}(N)$, such that $\text{MP}_\mu^{BS}(\Delta, \dots, \Delta) = \int^{BS} \Delta \mu$ for every convex body $\Delta \subset N$.*

MONOTONICITY OF MIXED FIBER BODIES. Proof of Theorem 6.8 gives the following fact as a byproduct.

THEOREM 6.30. *In the notation of Theorem 6.2, consider convex bodies $\Delta_0, \dots, \Delta_m, \Delta'_0, \dots, \Delta'_m$ in the space $L \oplus M$. If $\Delta_i \subset \Delta'_i$ and $v(\Delta_i) = v(\Delta'_i)$ for every i , where v is the projection $L \oplus M \rightarrow M$, then $\text{MP}_\mu(\Delta_0, \dots, \Delta_m) \subset \text{MP}_\mu(\Delta'_0, \dots, \Delta'_m)$.*

If $v(\Delta_i) \neq v(\Delta'_i)$, then the statement is not true in general, but $\text{MP}_\mu(\Delta_0, \dots, \Delta_m) \subset \text{MP}_\mu(\Delta'_0, \dots, \Delta'_m) + a$ for a suitable $a \in L$ (see [EKH]).

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